

# Complex Numbers in Three Dimensions

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## Abstract

A system of commutative hypercomplex numbers of the form  $w = x + hy + kz$  are introduced in 3 dimensions, the variables  $x, y$  and  $z$  being real numbers. The multiplication rules for the complex units  $h, k$  are  $h^2 = k, k^2 = h, hk = 1$ . The operations of addition and multiplication of the tricomplex numbers introduced in this paper have a simple geometric interpretation based on the modulus  $d$ , amplitude  $\rho$ , polar angle  $\theta$  and azimuthal angle  $\phi$ . Exponential and trigonometric forms are obtained for the tricomplex numbers, depending on the variables  $d, \rho, \theta$  and  $\phi$ . The tricomplex functions defined by series of powers are analytic, and the partial derivatives of the components of the tricomplex functions are closely related. The integrals of tricomplex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the tricomplex numbers contains the cyclic variable  $\phi$  leads to the concepts of pole and residue for integrals of tricomplex functions on closed paths. The polynomials of tricomplex variables can be written as products of linear or quadratic factors.

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# 1 Introduction

A regular, two-dimensional complex number  $x + iy$  can be represented geometrically by the modulus  $\rho = (x^2 + y^2)^{1/2}$  and by the polar angle  $\theta = \arctan(y/x)$ . The modulus  $\rho$  is multiplicative and the polar angle  $\theta$  is additive upon the multiplication of ordinary complex numbers.

The quaternions of Hamilton are a system of hypercomplex numbers defined in four dimensions, the multiplication being a noncommutative operation, [1] and many other hypercomplex systems are possible, [2]-[4] but these hypercomplex systems do not have all the required properties of regular, two-dimensional complex numbers which rendered possible the development of the theory of functions of a complex variable.

A system of hypercomplex numbers in three dimensions is described in this work, for which the multiplication is associative and commutative, and which is rich enough in properties so that exponential and trigonometric forms exist for these numbers, and the concepts of analytic tricomplex function, contour integration and residue can be defined. The tricomplex numbers introduced in this work have the form  $u = x + hy + kz$ , the variables  $x, y$  and  $z$  being real numbers. The multiplication rules for the complex units  $h, k$  are  $h^2 = k, k^2 = h, hk = 1$ . In a geometric representation, the tricomplex number  $u$  is represented by the point  $P$  of coordinates  $(x, y, z)$ . If  $O$  is the origin of the  $x, y, z$  axes,  $(t)$  the trisector line  $x = y = z$  of the positive octant and  $\Pi$  the plane  $x + y + z = 0$  passing through the origin ( $O$ ) and perpendicular to  $(t)$ , then the tricomplex number  $u$  can be described by the projection  $s$  of the segment  $OP$  along the line  $(t)$ , by the distance  $D$  from  $P$  to the line  $(t)$ , and by the azimuthal angle  $\phi$  of the projection of  $P$  on the plane  $\Pi$ , measured from an angular origin defined by the intersection of the plane determined by the line  $(t)$  and the  $x$  axis, with the plane  $\Pi$ . The amplitude  $\rho$  of a twocomplex number is defined as  $\rho = (x^3 + y^3 + z^3 - 3xyz)^{1/3}$ , the polar angle  $\theta$  of  $OP$  with respect to the trisector line  $(t)$  is given by  $\tan \theta = D/s$ , and  $d^2 = x^2 + y^2 + z^2$ . The amplitude  $\rho$  is equal to zero on the trisector line  $(t)$  and on the plane  $\Pi$ . The division  $1/(x + hy + kz)$  is possible provided that  $\rho \neq 0$ . The product of two tricomplex numbers is equal to zero if both numbers are equal to zero, or if one of the tricomplex numbers lies in the  $\Pi$  plane and the other on the  $(t)$  line.

If  $u_1 = x_1 + hy_1 + kz_1, u_2 = x_2 + hy_2 + kz_2$  are tricomplex numbers of amplitudes and angles

$\rho_1, \theta_1, \phi_1$  and respectively  $\rho_2, \theta_2, \phi_2$ , then the amplitude and the angles  $\rho, \theta, \phi$  for the product tricomplex number  $u_1 u_2 = x_1 x_2 + y_1 z_2 + y_2 z_1 + h(z_1 z_2 + x_1 y_2 + y_1 x_2) + k(y_1 y_2 + x_1 z_2 + z_1 x_2)$  are  $\rho = \rho_1 \rho_2, \tan \theta = \tan \theta_1 \tan \theta_2 / \sqrt{2}, \phi = \phi_1 + \phi_2$ . Thus, the amplitude  $\rho$  and  $(\tan \theta) / \sqrt{2}$  are multiplicative quantities and the angle  $\phi$  is an additive quantity upon the multiplication of tricomplex numbers, which reminds the properties of ordinary, two-dimensional complex numbers.

For the description of the exponential function of a tricomplex variable, it is useful to define the cosexponential functions  $\text{cx}(\xi) = 1 + \xi^3/3! + \xi^6/6! \dots, \text{mx}(\xi) = \xi + \xi^4/4! + \xi^7/7! \dots, \text{px}(\xi) = \xi^2/2 + \xi^5/5! + \xi^8/8! \dots$ , where p and m stand for plus and respectively minus, as a reference to the sign of a phase shift in the expressions of these functions. These functions fulfil the relation  $\text{cx}^3 \xi + \text{px}^3 \xi + \text{mx}^3 \xi - 3 \text{cx} \xi \text{px} \xi \text{mx} \xi = 1$ .

The exponential form of a tricomplex number is  $u = \rho \exp \left[ (1/3)(h+k) \ln(\sqrt{2}/\tan \theta) + (1/3)(h-k)\phi \right]$ , and the trigonometric form of the tricomplex number is  $u = d\sqrt{3}/2 \left\{ (1/3)(2-h-k) \sin \theta + (1/3)(1+h+k)\sqrt{2} \cos \theta \right\} \exp \left\{ (h-k)\phi/\sqrt{3} \right\}$ .

Expressions are given for the elementary functions of tricomplex variable. Moreover, it is shown that the region of convergence of series of powers of tricomplex variables are cylinders with the axis parallel to the trisector line. A function  $f(u)$  of the tricomplex variable  $u = x + hy + kz$  can be defined by a corresponding power series. It will be shown that the function  $f(u)$  has a derivative at  $u_0$  independent of the direction of approach of  $u$  to  $u_0$ . If the tricomplex function  $f(u)$  of the tricomplex variable  $u$  is written in terms of the real functions  $F(x, y, z), G(x, y, z), H(x, y, z)$  of real variables  $x, y, z$  as  $f(u) = F(x, y, z) + hG(x, y, z) + kH(x, y, z)$ , then relations of equality exist between partial derivatives of the functions  $F, G, H$ , and the differences  $F - G, F - H, G - H$  are solutions of the equation of Laplace.

It will be shown that the integral  $\int_A^B f(u) du$  of a regular tricomplex function between two points  $A, B$  is independent of the three-dimensional path connecting the points  $A, B$ . If  $f(u)$  is an analytic tricomplex function, then  $\oint_{\Gamma} f(u) du / (u - u_0) = 2\pi(h-k)f(u_0)$  if the integration loop is threaded by the parallel through  $u_0$  to the line  $(t)$ .

A tricomplex polynomial  $u^m + a_1 u^{m-1} + \dots + a_{m-1} u + a_m$  can be written as a product of linear or quadratic factors, although the factorization may not be unique.

This paper belongs to a series of studies on commutative complex numbers in  $n$  dimensions. [5] The tricomplex numbers described in this work are a particular case for  $n = 3$  of the polar hypercomplex numbers in  $n$  dimensions.[5],[6]

## 2 Operations with tricomplex numbers

A tricomplex number is determined by its three components  $(x, y, z)$ . The sum of the tricomplex numbers  $(x, y, z)$  and  $(x', y', z')$  is the tricomplex number  $(x + x', y + y', z + z')$ . The product of the tricomplex numbers  $(x, y, z)$  and  $(x', y', z')$  is defined in this work to be the tricomplex number  $(xx' + yz' + zy', zz' + xy' + yx', yy' + xz' + zx')$ .

Tricomplex numbers and their operations can be represented by writing the tricomplex number  $(x, y, z)$  as  $u = x + hy + kz$ , where  $h$  and  $k$  are bases for which the multiplication rules are

$$h^2 = k, k^2 = h, 1 \cdot h = h, 1 \cdot k = k, hk = 1. \quad (1)$$

Two tricomplex numbers  $u = x + hy + kz, u' = x' + hy' + kz'$  are equal,  $u = u'$ , if and only if  $x = x', y = y', z = z'$ . If  $u = x + hy + kz, u' = x' + hy' + kz'$  are tricomplex numbers, the sum  $u + u'$  and the product  $uu'$  defined above can be obtained by applying the usual algebraic rules to the sum  $(x + hy + kz) + (x' + hy' + kz')$  and to the product  $(x + hy + kz)(x' + hy' + kz')$ , and grouping of the resulting terms,

$$u + u' = x + x' + h(y + y') + k(z + z'), \quad (2)$$

$$uu' = xx' + yz' + zy' + h(zz' + xy' + yx') + k(yy' + xz' + zx'). \quad (3)$$

If  $u, u', u''$  are tricomplex numbers, the multiplication is associative

$$(uu')u'' = u(u'u'') \quad (4)$$

and commutative

$$uu' = u'u, \quad (5)$$

as can be checked through direct calculation. The tricomplex zero is  $0 + h \cdot 0 + k \cdot 0$ , denoted simply 0, and the tricomplex unity is  $1 + h \cdot 0 + k \cdot 0$ , denoted simply 1.

The inverse of the tricomplex number  $u = x + hy + kz$  is a tricomplex number  $u' = x' + y' + z'$  having the property that

$$uu' = 1. \quad (6)$$

Written on components, the condition, Eq. (6), is

$$\begin{aligned} xx' + zy' + yz' &= 1, \\ yx' + xy' + zz' &= 0, \\ zx' + yy' + xz' &= 0. \end{aligned} \quad (7)$$

The system (7) has the solution

$$x' = \frac{x^2 - yz}{x^3 + y^3 + z^3 - 3xyz}, \quad (8)$$

$$y' = \frac{z^2 - xy}{x^3 + y^3 + z^3 - 3xyz}, \quad (9)$$

$$z' = \frac{y^2 - xz}{x^3 + y^3 + z^3 - 3xyz}, \quad (10)$$

provided that  $x^3 + y^3 + z^3 - 3xyz \neq 0$ . Since

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz), \quad (11)$$

a tricomplex number  $x + hy + kz$  has an inverse, unless

$$x + y + z = 0 \quad (12)$$

or

$$x^2 + y^2 + z^2 - xy - xz - yz = 0. \quad (13)$$

The relation in Eq. (12) represents the plane  $\Pi$  perpendicular to the trisector line  $(t)$  of the  $x, y, z$  axes, and passing through the origin  $O$  of the axes. The plane  $\Pi$ , shown in Fig. 1, intersects the  $xOy$  plane along the line  $z = 0, x + y = 0$ , it intersects the  $yOz$  plane along the line  $x = 0, y + z = 0$ , and it intersects the  $xOz$  plane along the line  $y = 0, x + z = 0$ . The condition (13) is equivalent to  $(x - y)^2 + (x - z)^2 + (y - z)^2 = 0$ , which for real  $x, y, z$  means that  $x = y = z$ , which represents the trisector line  $(t)$  of the axes  $x, y, z$ . The trisector line  $(t)$  is perpendicular to the plane  $\Pi$ . Because of conditions (12) and (13), the trisector line  $(t)$  and the plane  $\Pi$  will be also called nodal line and respectively nodal plane.

It can be shown that if  $uu' = 0$  then either  $u = 0$ , or  $u' = 0$ , or one of the tricomplex numbers  $u, u'$  belongs to the trisector line  $(t)$  and the other belongs to the nodal plane  $\Pi$ .

### 3 Geometric representation of tricomplex numbers

The tricomplex number  $x + hy + kz$  can be represented by the point  $P$  of coordinates  $(x, y, z)$ .

If  $O$  is the origin of the axes, then the projection  $s = OQ$  of the line  $OP$  on the trisector line  $x = y = z$ , which has the unit tangent  $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ , is

$$s = \frac{1}{\sqrt{3}}(x + y + z). \quad (14)$$

The distance  $D = PQ$  from  $P$  to the trisector line  $x = y = z$ , calculated as the distance from the point  $(x, y, z)$  to the point  $Q$  of coordinates  $[(x + y + z)/3, (x + y + z)/3, (x + y + z)/3]$ , is

$$D^2 = \frac{2}{3}(x^2 + y^2 + z^2 - xy - xz - yz). \quad (15)$$

The quantities  $s$  and  $D$  are shown in Fig. 2, where the plane through the point  $P$  and perpendicular to the trisector line  $(t)$  intersects the  $x$  axis at point  $A$  of coordinates  $(x + y + z, 0, 0)$ , the  $y$  axis at point  $B$  of coordinates  $(0, x + y + z, 0)$ , and the  $z$  axis at point  $C$  of coordinates  $(0, 0, x + y + z)$ . The azimuthal angle  $\phi$  of the tricomplex number  $x + hy + kz$  is defined as the angle in the plane  $\Pi$  of the projection of  $P$  on this plane, measured from the line of intersection of the plane determined by the line  $(t)$  and the  $x$  axis with the plane  $\Pi$ ,  $0 \leq \phi < 2\pi$ . The expression of  $\phi$  in terms of  $x, y, z$  can be obtained in a system of coordinates defined by the unit vectors

$$\xi_1 : \frac{1}{\sqrt{6}}(2, -1, -1); \xi_2 : \frac{1}{\sqrt{2}}(0, 1, -1); \xi_3 : \frac{1}{\sqrt{3}}(1, 1, 1), \quad (16)$$

and having the point  $O$  as origin. The relation between the coordinates of  $P$  in the systems  $\xi_1, \xi_2, \xi_3$  and  $x, y, z$  can be written in the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (17)$$

The components of the vector  $OP$  in the system  $\xi_1, \xi_2, \xi_3$  can be obtained with the aid of Eq. (17) as

$$(\xi_1, \xi_2, \xi_3) = \left( \frac{1}{\sqrt{6}}(2x - y - z), \frac{1}{\sqrt{2}}(y - z), \frac{1}{\sqrt{3}}(x + y + z) \right). \quad (18)$$

The expression of the angle  $\phi$  as a function of  $x, y, z$  is then

$$\cos \phi = \frac{2x - y - z}{2(x^2 + y^2 + z^2 - xy - xz - yz)^{1/2}}, \quad (19)$$

$$\sin \phi = \frac{\sqrt{3}(y-z)}{2(x^2 + y^2 + z^2 - xy - xz - yz)^{1/2}}. \quad (20)$$

It can be seen from Eqs. (19),(20) that the angle of points on the  $x$  axis is  $\phi = 0$ , the angle of points on the  $y$  axis is  $\phi = 2\pi/3$ , and the angle of points on the  $z$  axis is  $\phi = 4\pi/3$ . The angle  $\phi$  is shown in Fig. 2 in the plane parallel to  $\Pi$ , passing through  $P$ . The axis  $Q\xi_1^{\parallel}$  is parallel to the axis  $O\xi_1$ , the axis  $Q\xi_2^{\parallel}$  is parallel to the axis  $O\xi_2$ , and the axis  $Q\xi_3^{\parallel}$  is parallel to the axis  $O\xi_3$ , so that, in the plane  $ABC$ , the angle  $\phi$  is measured from the line  $QA$ . The angle  $\theta$  between the line  $OP$  and the trisector line ( $t$ ) is given by

$$\tan \theta = \frac{D}{s}, \quad (21)$$

where  $0 \leq \theta \leq \pi$ . It can be checked that

$$d^2 = D^2 + s^2, \quad (22)$$

where

$$d^2 = x^2 + y^2 + z^2, \quad (23)$$

so that

$$D = d \sin \theta, \quad s = d \cos \theta. \quad (24)$$

The relations (14), (15), (19)-(21) can be used to determine the associated projection  $s$ , the distance  $D$ , the polar angle  $\theta$  with the trisector line ( $t$ ) and the angle  $\phi$  in the  $\Pi$  plane for the tricomplex number  $x + hy + kz$ . It can be shown that if  $u_1 = x_1 + hy_1 + kz_1, u_2 = x_2 + hy_2 + kz_2$  are tricomplex numbers of projections, distances and angles  $s_1, D_1, \theta_1, \phi_1$  and respectively  $s_2, D_2, \theta_2, \phi_2$ , then the projection  $s$ , distance  $D$  and the angle  $\theta, \phi$  for the product tricomplex number  $u_1 u_2 = x_1 x_2 + y_1 z_2 + y_2 z_1 + h(z_1 z_2 + x_1 y_2 + y_1 x_2) + k(y_1 y_2 + x_1 z_2 + z_1 x_2)$  are

$$s = \sqrt{3}s_1 s_2, \quad D = \sqrt{\frac{3}{2}}D_1 D_2, \quad \tan \theta = \frac{1}{\sqrt{2}} \tan \theta_1 \tan \theta_2, \quad \phi = \phi_1 + \phi_2. \quad (25)$$

The relations (25) are consequences of the identities

$$\begin{aligned} & (x_1 x_2 + y_1 z_2 + y_2 z_1) + (z_1 z_2 + x_1 y_2 + y_1 x_2) + (y_1 y_2 + x_1 z_2 + z_1 x_2) \\ & = (x_1 + y_1 + z_1)(x_2 + y_2 + z_2), \end{aligned} \quad (26)$$

$$\begin{aligned}
& (x_1x_2 + y_1z_2 + y_2z_1)^2 + (z_1z_2 + x_1y_2 + y_1x_2)^2 + (y_1y_2 + x_1z_2 + z_1x_2)^2 \\
& - (x_1x_2 + y_1z_2 + y_2z_1)(z_1z_2 + x_1y_2 + y_1x_2) - (x_1x_2 + y_1z_2 + y_2z_1)(y_1y_2 + x_1z_2 + z_1x_2) \\
& - (z_1z_2 + x_1y_2 + y_1x_2) + (y_1y_2 + x_1z_2 + z_1x_2) \\
& = (x_1^2 + y_1^2 + z_1^2 - x_1y_1 - x_1z_1 - y_1z_1)(x_2^2 + y_2^2 + z_2^2 - x_2y_2 - x_2z_2 - y_2z_2), \tag{27}
\end{aligned}$$

$$\begin{aligned}
& \frac{2x_1 - y_1 - z_1}{2} \frac{2x_2 - y_2 - z_2}{2} - \frac{\sqrt{3}}{2}(y_1 - z_1) \frac{\sqrt{3}}{2}(y_2 - z_2) \\
& = \frac{1}{2}[2(x_1x_2 + y_1z_2 + z_1y_2) - (z_1z_2 + x_1y_2 + y_1x_2) - (y_1y_2 + x_1z_2 + z_1x_2)], \tag{28}
\end{aligned}$$

$$\begin{aligned}
& \frac{\sqrt{3}}{2}(y_1 - z_1) \frac{2x_2 - y_2 - z_2}{2} + \frac{\sqrt{3}}{2}(y_2 - z_2) \frac{2x_1 - y_1 - z_1}{2} \\
& = \frac{\sqrt{3}}{2}[(z_1z_2 + x_1y_2 + y_1x_2) - (y_1y_2 + x_1z_2 + z_1x_2)]. \tag{29}
\end{aligned}$$

The relation (26) shows that if  $u$  is in the plane  $\Pi$ , such that  $x + y + z = 0$ , then the product  $uu'$  is also in the plane  $\Pi$  for any  $u'$ . The relation (27) shows that if  $u$  is on the trisector line  $(t)$ , such that  $x^2 + y^2 + z^2 - xy - xz - yz = 0$ , then  $uu'$  is also on the trisector line  $(t)$  for any  $u'$ . If  $u, u'$  are points in the plane  $x + y + z = 1$ , then the product  $uu'$  is also in that plane, and if  $u, u'$  are points of the cylindrical surface  $x^2 + y^2 + z^2 - xy - xz - yz = 1$ , then  $uu'$  is also in that cylindrical surface. This means that if  $u, u'$  are points on the circle  $x + y + z = 1, x^2 + y^2 + z^2 - xy - xz - yz = 1$ , which is perpendicular to the trisector line, is situated at a distance  $1/\sqrt{3}$  from the origin and has the radius  $\sqrt{2/3}$ , then the tricomplex product  $uu'$  is also on the same circle. This invariant circle for the multiplication of tricomplex numbers is described by the equations

$$x = \frac{1}{3} + \frac{2}{3} \cos \phi, \quad y = \frac{1}{3} - \frac{1}{3} \cos \phi + \frac{1}{\sqrt{3}} \sin \phi, \quad z = \frac{1}{3} - \frac{1}{3} \cos \phi - \frac{1}{\sqrt{3}} \sin \phi. \tag{30}$$

It has the center at the point  $(1/3, 1/3, 1/3)$  and passes through the points  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ , as shown in Fig. 3.

An important quantity is the amplitude  $\rho$  defined as  $\rho = \nu^{1/3}$ , so that

$$\rho^3 = x^3 + y^3 + z^3 - 3xyz. \tag{31}$$

The amplitude  $\rho$  of the product  $u_1 u_2$  of the tricomplex numbers  $u_1, u_2$  of amplitudes  $\rho_1, \rho_2$  is

$$\rho = \rho_1 \rho_2, \tag{32}$$



as can be seen from the identity

$$\begin{aligned}
& (x_1x_2 + y_1z_2 + y_2z_1)^3 + (z_1z_2 + x_1y_2 + y_1x_2)^3 + (y_1y_2 + x_1z_2 + z_1x_2)^3 \\
& - 3(x_1x_2 + y_1z_2 + y_2z_1)(z_1z_2 + x_1y_2 + y_1x_2)(y_1y_2 + x_1z_2 + z_1x_2) \\
& = (x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1)(x_2^3 + y_2^3 + z_2^3 - 3x_2y_2z_2).
\end{aligned} \tag{33}$$

The identity in Eq. (33) can be demonstrated with the aid of Eqs. (11), (26) and (27). Another method would be to use the representation of the multiplication of the tricomplex numbers by matrices, in which the tricomplex number  $u = x + hy + kz$  is represented by the matrix

$$\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}. \tag{34}$$

The product  $u = x + hy + kz$  of the tricomplex numbers  $u_1 = x_1 + hy_1 + kz_1$ ,  $u_2 = x_2 + hy_2 + kz_2$ , is represented by the matrix multiplication

$$\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ z_1 & x_1 & y_1 \\ y_1 & z_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & y_2 & z_2 \\ z_2 & x_2 & y_2 \\ y_2 & z_2 & x_2 \end{pmatrix}. \tag{35}$$

If

$$\nu = \det \begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}, \tag{36}$$

it can be checked that

$$\nu = x^3 + y^3 + z^3 - 3xyz. \tag{37}$$

The identity (33) is then a consequence of the fact the determinant of the product of matrices is equal to the product of the determinants of the factor matrices.

It can be seen from Eqs. (14) and (15) that

$$x^3 + y^3 + z^3 - 3xyz = \frac{3\sqrt{3}}{2} sD^2, \tag{38}$$

which can be written with the aid of relations (24) and (31) as

$$\rho = \frac{3^{1/2}}{2^{1/3}} d \sin^{2/3} \theta \cos^{1/3} \theta. \quad (39)$$

This means that the surfaces of constant  $\rho$  are surfaces of rotation having the trisector line  $(t)$  as axis, as shown in Fig. 4.

## 4 The tricomplex cosexponential functions

The exponential function of the tricomplex variable  $u$  can be defined by the series

$$\exp u = 1 + u + u^2/2! + u^3/3! + \dots \quad (40)$$

It can be checked by direct multiplication of the series that

$$\exp(u + u') = \exp u \cdot \exp u', \quad (41)$$

which is valid as long as the multiplication is a commutative operation. If  $u = x + hy + kz$ , then  $\exp u$  can be calculated as  $\exp u = \exp x \cdot \exp(hy) \cdot \exp(kz)$ . According to Eq. (1),  $h^2 = k, h^3 = 1, k^2 = h, k^3 = 1$ , and in general

$$h^{3m} = 1, h^{3m+1} = h, h^{3m+2} = k, k^{3m} = 1, k^{3m+1} = k, k^{3m+2} = h, \quad (42)$$

where  $n$  is a natural number, so that  $\exp(hy)$  and  $\exp(kz)$  can be written as

$$\exp(hy) = \text{cx } y + h \text{mx } y + k \text{px } y, \quad (43)$$

$$\exp(kz) = \text{cx } z + h \text{px } z + k \text{mx } z, \quad (44)$$

where the functions  $\text{cx}$ ,  $\text{mx}$ ,  $\text{px}$ , which will be called in this work polar cosexponential functions, are defined by the series

$$\text{cx } y = 1 + y^3/3! + y^6/6! + \dots \quad (45)$$

$$\text{mx } y = y + y^4/4! + y^7/7! + \dots \quad (46)$$

$$\text{px } y = y^2/2! + y^5/5! + y^8/8! + \dots \quad (47)$$

From the series definitions it can be seen that  $\text{cx}0 = 1, \text{mx}0 = 0, \text{px}0 = 0$ . The tridimensional polar cosexponential functions belong to the class of the polar  $n$ -dimensional cosexponential functions  $g_{nk}$ , [6] and  $\text{cx} = g_{30}, \text{mx} = g_{31}, \text{px} = g_{32}$ . It can be checked that

$$\text{cx } y + \text{px } y + \text{mx } y = \exp y. \quad (48)$$

By expressing the fact that  $\exp(hy+hz) = \exp(hy) \cdot \exp(hz)$  with the aid of the cosexponential functions (45)-(47) the following addition theorems can be obtained

$$\text{cx } (y + z) = \text{cx } y \text{ cx } z + \text{mx } y \text{ px } z + \text{px } y \text{ mx } z, \quad (49)$$

$$\text{mx } (y + z) = \text{px } y \text{ px } z + \text{cx } y \text{ mx } z + \text{mx } y \text{ cx } z, \quad (50)$$

$$\text{px } (y + z) = \text{mx } y \text{ mx } z + \text{cx } y \text{ px } z + \text{px } y \text{ cx } z. \quad (51)$$

For  $y = z$ , Eqs. (49)-(51) yield

$$\text{cx } 2y = \text{cx}^2 y + 2 \text{mx } y \text{ px } z, \quad (52)$$

$$\text{mx } 2y = \text{px}^2 y + 2 \text{cx } y \text{ mx } z, \quad (53)$$

$$\text{px } 2y = \text{mx}^2 y + 2 \text{cx } y \text{ px } z. \quad (54)$$

The cosexponential functions are neither even nor odd functions. For  $z = -y$ , Eqs. (49)-(51) yield

$$\text{cx } y \text{ cx } (-y) + \text{mx } y \text{ px } (-y) + \text{px } y \text{ mx } (-y) = 1, \quad (55)$$

$$\text{px } y \text{ px } (-y) + \text{cx } y \text{ mx } (-y) + \text{mx } y \text{ cx } (-y) = 0, \quad (56)$$

$$\text{mx } y \text{ mx } (-y) + \text{cx } y \text{ px } (-y) + \text{px } y \text{ cx } (-y) = 0. \quad (57)$$

Expressions of the cosexponential functions in terms of regular exponential and cosine functions can be obtained by considering the series expansions for  $e^{(h+k)y}$  and  $e^{(h-k)y}$ . These expressions can be obtained by calculating first  $(h+k)^n$  and  $(h-k)^n$ . It can be shown that

$$(h+k)^m = \frac{1}{3} \left[ (-1)^{m-1} + 2^m \right] (h+k) + \frac{2}{3} \left[ (-1)^m + 2^{m-1} \right], \quad (58)$$

$$(h-k)^{2m} = (-1)^{m-1} 3^{m-1} (k+k-2), \quad (h-k)^{2m+1} = (-1)^m 3^m (h-k), \quad (59)$$

where  $n$  is a natural number. Then

$$e^{(h+k)y} = (h+k) \left( -\frac{1}{3}e^{-y} + \frac{1}{3}e^{2y} \right) + \frac{2}{3}e^{-y} + \frac{1}{3}e^{2y}. \quad (60)$$

As a corollary, the following identities can be obtained from Eq. (60) by writing  $e^{(h+k)y} = e^{hy}e^{ky}$  and expressing  $e^{hy}$  and  $e^{ky}$  in terms of cosexponential functions via Eqs. (43) and (44),

$$cx^2 y + mx^2 y + px^2 y = \frac{2}{3}e^{-y} + \frac{1}{3}e^{2y}, \quad (61)$$

$$cx y \ mx y + cx y \ px y + mx y \ px y = -\frac{1}{3}e^{-y} + \frac{1}{3}e^{2y}. \quad (62)$$

From Eqs. (61) and (62) it results that

$$\begin{aligned} & cx^2 y + mx^2 y + px^2 y \\ & - cx y \ mx y - cx y \ px y - mx y \ px y = \exp(-y). \end{aligned} \quad (63)$$

Then from Eqs. (11), (48) and (63) it follows that

$$cx^3 y + mx^3 y + px^3 y - 3cx y \ mx y \ px y = 1. \quad (64)$$

Similarly,

$$e^{(h-k)y} = \frac{1}{3}(1+h+k) + \frac{1}{3}(2-h-k)\cos(\sqrt{3}y) + \frac{1}{\sqrt{3}}(h-k)\sin(\sqrt{3}y). \quad (65)$$

The last relation can also be written as

$$\begin{aligned} e^{(h-k)y} &= \frac{1}{3} + \frac{2}{3}\cos(\sqrt{3}y) + h \left[ \frac{1}{3} + \frac{2}{3}\cos\left(\sqrt{3}y - \frac{2\pi}{3}\right) \right] \\ &+ k \left[ \frac{1}{3} + \frac{2}{3}\cos\left(\sqrt{3}y + \frac{2\pi}{3}\right) \right]. \end{aligned} \quad (66)$$

As a corollary, the following identities can be obtained from Eq. (65) by writing  $e^{(h-k)y} = e^{hy}e^{-ky}$  and expressing  $e^{hy}$  and  $e^{-ky}$  in terms of cosexponential functions via Eqs. (43) and (44),

$$cx y \ cx (-y) + mx y \ mx (-y) + px y \ px (-y) = \frac{1}{3} + \frac{2}{3}\cos(\sqrt{3}y), \quad (67)$$

$$cx y \ px (-y) + mx y \ cx (-y) + px y \ mx (-y) = \frac{1}{3} + \frac{2}{3}\cos\left(\sqrt{3}y - \frac{2\pi}{3}\right) \quad (68)$$

$$\text{cx } y \text{ mx } (-y) + \text{mx } y \text{ px } (-y) + \text{px } y \text{ cx } (-y) = \frac{1}{3} + \frac{2}{3} \cos \left( \sqrt{3}y + \frac{2\pi}{3} \right) \quad (69)$$

Expressions of  $e^{2hy}$  in terms of the regular exponential and cosine functions can be obtained by the multiplication of the expressions of  $e^{(h+k)y}$  and  $e^{(h-k)y}$  from Eqs. (60) and (65). At the same time, Eq. (43) gives an expression of  $e^{2hy}$  in terms of cosexponential functions. By equating the real and hypercomplex parts of these two forms of  $e^{2y}$  and then replacing  $2y$  by  $y$  gives the expressions of the cosexponential functions as

$$\text{cx } y = \frac{1}{3} e^y + \frac{2}{3} \cos \left( \frac{\sqrt{3}}{2} y \right) e^{-y/2}, \quad (70)$$

$$\text{mx } y = \frac{1}{3} e^y + \frac{2}{3} \cos \left( \frac{\sqrt{3}}{2} y - \frac{2\pi}{3} \right) e^{-y/2}, \quad (71)$$

$$\text{px } y = \frac{1}{3} e^y + \frac{2}{3} \cos \left( \frac{\sqrt{3}}{2} y + \frac{2\pi}{3} \right) e^{-y/2}. \quad (72)$$

It is remarkable that the series in Eqs. (45)-(47), in which the terms are either of the form  $y^{3m}$ , or  $y^{3m+1}$ , or  $y^{3m+2}$ , can be expressed in terms of elementary functions whose power series are not subject to such restrictions. The cosexponential functions differ by the phase of the cosine function in their expression, and the designation of the functions in Eqs. (71) and (72) as mx and px refers respectively to the minus or plus sign of the phase term  $2\pi/3$ . The graphs of the cosexponential functions are shown in Fig. 5.

It can be checked that the cosexponential functions are solutions of the third-order differential equation

$$\frac{d^3 \zeta}{du^3} = \zeta, \quad (73)$$

whose solutions are of the form  $\zeta(u) = A \text{ cx } u + B \text{ mx } u + C \text{ px } u$ . It can also be checked that the derivatives of the cosexponential functions are related by

$$\frac{d \text{ px}}{du} = \text{mx}, \quad \frac{d \text{ mx}}{du} = \text{cx}, \quad \frac{d \text{ cx}}{du} = \text{px}. \quad (74)$$

## 5 Exponential and trigonometric forms of tricomplex numbers

If for a tricomplex number  $u = x + ky + kz$  another tricomplex number  $u_1 = x_1 + hy_1 + kz_1$  exists such that

$$x + hy + kz = e^{x_1 + hy_1 + kz_1}, \quad (75)$$

then  $u_1$  is said to be the logarithm of  $u$ ,

$$u_1 = \ln u. \quad (76)$$

The expressions of  $x_1, y_1, z_1$  as functions of  $x, y, z$  can be obtained by developing  $e^{hy_1}$  and  $e^{kz_1}$  with the aid of Eqs. (43) and (44), by multiplying these expressions and separating the hypercomplex components,

$$x = e^{x_1} [\text{cx } y_1 \text{ cx } z_1 + \text{mx } y_1 \text{ mx } z_1 + \text{px } y_1 \text{ px } z_1], \quad (77)$$

$$y = e^{x_1} [\text{cx } y_1 \text{ px } z_1 + \text{mx } y_1 \text{ cx } z_1 + \text{px } y_1 \text{ mx } z_1], \quad (78)$$

$$z = e^{x_1} [\text{cx } y_1 \text{ mx } z_1 + \text{px } y_1 \text{ cx } z_1 + \text{mx } y_1 \text{ px } z_1], \quad (79)$$

Using Eq. (33) with the substitutions  $x_1 \rightarrow \text{cx } y_1, y_1 \rightarrow \text{mx } y_1, z_1 \rightarrow \text{px } y_1, x_2 \rightarrow \text{cx } z_1, y_2 \rightarrow \text{px } z_1, z_2 \rightarrow \text{mx } z_1$  and then the identity (64) yields

$$x^3 + y^3 + z^3 - 3xyz = e^{3x_1}, \quad (80)$$

whence

$$x_1 = \frac{1}{3} \ln(x^3 + y^3 + z^3 - 3xyz). \quad (81)$$

The logarithm in Eq. (81) exists as a real function for  $x + y + z > 0$ . A further relation can be obtained by summing Eqs. (77)-(79) and then using the addition theorems (49)-(51)

$$\frac{x + y + z}{(x^3 + y^3 + z^3 - 3xyz)^{1/3}} = \text{cx } (y_1 + z_1) + \text{mx } (y_1 + z_1) + \text{px } (y_1 + z_1). \quad (82)$$

The sum in Eq. (82) is according to Eq. (48)  $e^{y_1 + z_1}$ , so that

$$y_1 + z_1 = \ln \frac{x + y + z}{(x^3 + y^3 + z^3 - 3xyz)^{1/3}}. \quad (83)$$

The logarithm in Eq. (83) is defined for points which are not on the trisector line ( $t$ ), so that  $x^2 + y^2 + z^2 - xy - xz - yz \neq 0$ . Substituting in Eq. (75) the expression of  $x_1$ , Eq. (81), and of  $z_1$  as a function of  $x, y, z, y_1$ , Eq. (83), yields

$$\frac{u}{\rho} \exp \left[ -k \ln \left( \frac{\sqrt{2}s}{D} \right)^{2/3} \right] = e^{(h-k)y_1}, \quad (84)$$

where the quantities  $\rho, s$  and  $D$  have been defined in Eqs. (31), (14) and (15). Developing the exponential functions in the left-hand side of Eq. (84) with the aid of Eq. (44) and using the expressions of the cosexponential functions, Eqs. (70)-(72), and using the relation (65) for the right-hand side of Eq. (84) yields for the real part

$$\frac{\left( x - \frac{y+z}{2} \right) \cos \left[ \frac{1}{\sqrt{3}} \ln \left( \frac{\sqrt{2}s}{D} \right) \right] - \frac{\sqrt{3}}{2} (y - z) \sin \left[ \frac{1}{\sqrt{3}} \ln \left( \frac{\sqrt{2}s}{D} \right) \right]}{(x^2 + y^2 + z^2 - xy - xz - yz)^{1/2}} = \cos(\sqrt{3}y_1), \quad (85)$$

which can also be written as

$$\cos \left[ \frac{1}{\sqrt{3}} \ln \left( \frac{\sqrt{2}s}{D} \right) + \phi \right] = \cos(\sqrt{3}y_1) \quad (86)$$

where  $\phi$  is the angle defined in Eqs. (19) and (20). Thus

$$y_1 = \frac{1}{3} \ln \left( \frac{\sqrt{2}s}{D} \right) + \frac{1}{\sqrt{3}} \phi. \quad (87)$$

The exponential form of the tricomplex number  $u$  is then

$$u = \rho \exp \left[ \frac{1}{3} (h + k) \ln \frac{\sqrt{2}}{\tan \theta} + \frac{1}{\sqrt{3}} (h - k) \phi \right], \quad (88)$$

where  $\theta$  is the angle between the line  $OP$  connecting the origin to the point  $P$  of coordinates  $(x, y, z)$  and the trisector line ( $t$ ), defined in Eq. (21) and shown in Fig. 2. The exponential in Eq. (88) can be expanded with the aid of Eq. (60) and (66) as

$$\exp \left[ \frac{1}{3} (h + k) \ln \frac{\sqrt{2}}{\tan \theta} \right] = \frac{2 - h - k}{3} \left( \frac{\tan \theta}{\sqrt{2}} \right)^{1/3} + \frac{1 + h + k}{3} \left( \frac{\sqrt{2}}{\tan \theta} \right)^{2/3}, \quad (89)$$

so that

$$x + hy + kz = \rho \left[ \frac{2 - h - k}{3} \left( \frac{\tan \theta}{\sqrt{2}} \right)^{1/3} + \frac{1 + h + k}{3} \left( \frac{\sqrt{2}}{\tan \theta} \right)^{2/3} \right] \exp \left( \frac{h - k}{\sqrt{3}} \phi \right). \quad (90)$$

Substituting in Eq. (90) the expression of the amplitude  $\rho$ , Eq. (39), yields

$$u = d \sqrt{\frac{3}{2}} \left( \frac{2 - h - k}{3} \sin \theta + \frac{1 + h + k}{3} \sqrt{2} \cos \theta \right) \exp \left( \frac{h - k}{\sqrt{3}} \phi \right), \quad (91)$$

which is the trigonometric form of the tricomplex number  $u$ . As can be seen from Eq. (91), the tricomplex number  $x + hy + kz$  is written as the product of the modulus  $d$ , of a factor depending on the polar angle  $\theta$  with respect to the trisector line, and of a factor depending of the azimuthal angle  $\phi$  in the plane  $\Pi$  perpendicular to the trisector line. The exponential in Eq. (91) can be expanded further with the aid of Eq. (66) as

$$\exp\left(\frac{1}{\sqrt{3}}(h-k)\phi\right) = \frac{1+h+k}{3} + \frac{2-h-k}{3}\cos\phi + \frac{h-k}{\sqrt{3}}\sin\phi, \quad (92)$$

so that the tricomplex number  $x + hy + kz$  can also be written, after multiplication of the factors, in the form

$$\begin{aligned} x + hy + kz = & \frac{2-h-k}{3}(x^2 + y^2 + z^2 - xy - xz - yz)^{1/2} \cos\phi \\ & + \frac{h-k}{\sqrt{3}}(x^2 + y^2 + z^2 - xy - xz - yz)^{1/2} \sin\phi + \frac{1+h+k}{3}(x + y + z) \end{aligned} \quad (93)$$

The validity of Eq. (93) can be checked by substituting the expressions of  $\cos\phi$  and  $\sin\phi$  from Eqs. (19) and (20).

## 6 Elementary functions of a tricomplex variable

It can be shown with the aid of Eq. (88) that

$$(x+hy+kz)^m = \rho^m \left[ \frac{2-h-k}{3} \left( \frac{\tan\theta}{\sqrt{2}} \right)^{m/3} + \frac{1+h+k}{3} \left( \frac{\sqrt{2}}{\tan\theta} \right)^{2m/3} \right] \exp\left(\frac{h-k}{\sqrt{3}}m\phi\right), \quad (94)$$

or equivalently

$$\begin{aligned} (x + hy + kz)^m = & \frac{2-h-k}{3}(x^2 + y^2 + z^2 - xy - xz - yz)^{m/2} \cos(m\phi) \\ & + \frac{h-k}{\sqrt{3}}(x^2 + y^2 + z^2 - xy - xz - yz)^{m/2} \sin(m\phi) + \frac{1+h+k}{3}(x + y + z)^m \end{aligned} \quad (95)$$

which are valid for real values of  $m$ . Thus Eqs. (94) or (95) define the power function  $u^m$  of the tricomplex variable  $u$ .

The power function is multivalued unless  $m$  is an integer. It can be inferred from Eq. (88) that, for integer values of  $m$ ,

$$(uu')^m = u^m u'^m. \quad (96)$$



For natural  $m$ , Eq. (95) can be checked with the aid of relations (19) and (20). For example if  $m = 2$ , it can be checked that the right-hand side of Eq. (95) is equal to  $(x + hy + kz)^2 = x^2 + 2yz + h(z^2 + 2xz) + k(y^2 + 2xz)$ .

The logarithm  $u_1$  of the tricomplex number  $u$ ,  $u_1 = \ln u$ , can be defined as the solution of Eq. (75) for  $u_1$  as a function of  $u$ . For  $x + y + z > 0$ , from Eq. (88) it results that

$$\ln u = \ln \rho + \frac{1}{3}(h + k) \ln \left( \frac{\tan \theta}{\sqrt{2}} \right) + \frac{1}{\sqrt{3}}(h - k)\phi. \quad (97)$$

It can be checked with the aid of Eqs. (25) and (32) that

$$\ln(uu') = \ln u + \ln u', \quad (98)$$

which is valid up to integer multiples of  $2\pi(h - k)/\sqrt{3}$ .

The trigonometric functions  $\cos u$  and  $\sin u$  of the tricomplex variable  $u$  are defined by the series

$$\cos u = 1 - u^2/2! + u^4/4! + \dots, \quad (99)$$

$$\sin u = u - u^3/3! + u^5/5! + \dots. \quad (100)$$

It can be checked by series multiplication that the usual addition theorems hold also for the tricomplex numbers  $u, u'$ ,

$$\cos(u + u') = \cos u \cos u' - \sin u \sin u', \quad (101)$$

$$\sin(u + u') = \sin u \cos u' + \cos u \sin u'. \quad (102)$$

The trigonometric functions of the hypercomplex variables  $hy, ky$  can be expressed in terms of the cosexponential functions as

$$\cos(hy) = \frac{1}{2}[\text{cx}(iy) + \text{cx}(-iy)] + \frac{1}{2}h[\text{mx}(iy) + \text{mx}(-iy)] + \frac{1}{2}k[\text{px}(iy) + \text{px}(-iy)], \quad (103)$$

$$\cos(ky) = \frac{1}{2}[\text{cx}(iy) + \text{cx}(-iy)] + \frac{1}{2}h[\text{px}(iy) + \text{px}(-iy)] + \frac{1}{2}k[\text{mx}(iy) + \text{mx}(-iy)], \quad (104)$$

$$\sin(hy) = \frac{1}{2i}[\text{cx}(iy) - \text{cx}(-iy)] + \frac{1}{2i}h[\text{mx}(iy) - \text{mx}(-iy)] + \frac{1}{2i}k[\text{px}(iy) - \text{px}(-iy)], \quad (105)$$

$$\sin(ky) = \frac{1}{2i}[\text{cx}(iy) - \text{cx}(-iy)] + \frac{1}{2i}h[\text{px}(iy) - \text{px}(-iy)] + \frac{1}{2i}k[\text{mx}(iy) - \text{mx}(-iy)], \quad (106)$$

where  $i$  is the imaginary unit. Using the expressions of the cosexponential functions in Eqs. (70)-(72) gives expressions of the trigonometric functions of  $hy, hz$  as

$$\begin{aligned}\cos(hy) &= \frac{1}{3} \cos y + \frac{2}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} + \\ &+ h \left[ \frac{1}{3} \cos y - \frac{1}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} + \frac{1}{\sqrt{3}} \sinh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} \right] \\ &+ k \left[ \frac{1}{3} \cos y - \frac{1}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} - \frac{1}{\sqrt{3}} \sinh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} \right]\end{aligned}\quad (107)$$

$$\begin{aligned}\cos(ky) &= \frac{1}{3} \cos y + \frac{2}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} + \\ &+ h \left[ \frac{1}{3} \cos y - \frac{1}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} - \frac{1}{\sqrt{3}} \sinh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} \right] \\ &+ k \left[ \frac{1}{3} \cos y - \frac{1}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} + \frac{1}{\sqrt{3}} \sinh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} \right]\end{aligned}\quad (108)$$

$$\begin{aligned}\sin(hy) &= \frac{1}{3} \sin y - \frac{2}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} \\ &+ h \left[ \frac{1}{3} \sin y + \frac{1}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} + \frac{1}{\sqrt{3}} \sinh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} \right] \\ &+ k \left[ \frac{1}{3} \sin y + \frac{1}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} - \frac{1}{\sqrt{3}} \sinh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} \right]\end{aligned}\quad (109)$$

$$\begin{aligned}\sin(ky) &= \frac{1}{3} \sin y - \frac{2}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} \\ &+ h \left[ \frac{1}{3} \sin y + \frac{1}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} - \frac{1}{\sqrt{3}} \sinh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} \right] \\ &+ k \left[ \frac{1}{3} \sin y + \frac{1}{3} \cosh\left(\frac{\sqrt{3}}{2}y\right) \sin \frac{y}{2} + \frac{1}{\sqrt{3}} \sinh\left(\frac{\sqrt{3}}{2}y\right) \cos \frac{y}{2} \right]\end{aligned}\quad (110)$$

The trigonometric functions of a tricomplex number  $x + hy + kz$  can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (101), (102) and of the expressions in Eqs. (107)-(110).

The hyperbolic functions  $\cosh u$  and  $\sinh u$  of the fourcomplex variable  $u$  are defined by the series

$$\cosh u = 1 + u^2/2! + u^4/4! + \dots, \quad (111)$$

$$\sinh u = u + u^3/3! + u^5/5! + \dots \quad (112)$$

It can be checked by series multiplication that the usual addition theorems hold also for the fourcomplex numbers  $u, u'$ ,

$$\cosh(u + u') = \cosh u \cosh u' + \sinh u \sinh u', \quad (113)$$

$$\sinh(u + u') = \sinh u \cosh u' + \cosh u \sinh u'. \quad (114)$$

The hyperbolic functions of the hypercomplex variables  $hy, ky$  can be expressed in terms of the elementary functions as

$$\begin{aligned} \cosh(hy) &= \frac{1}{3} \cosh y + \frac{2}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} + \\ &+ h \left[ \frac{1}{3} \cosh y - \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} \right] \\ &+ k \left[ \frac{1}{3} \cosh y - \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} \right] \end{aligned} \quad (115)$$

$$\begin{aligned} \cosh(ky) &= \frac{1}{3} \cosh y + \frac{2}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} + \\ &+ h \left[ \frac{1}{3} \cosh y - \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} \right] \\ &+ k \left[ \frac{1}{3} \cosh y - \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} \right] \end{aligned} \quad (116)$$

$$\begin{aligned} \sinh(hy) &= \frac{1}{3} \sinh y - \frac{2}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} \\ &+ h \left[ \frac{1}{3} \sinh y + \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} \right] \\ &+ k \left[ \frac{1}{3} \sinh y + \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} \right] \end{aligned} \quad (117)$$

$$\begin{aligned} \sinh(ky) &= \frac{1}{3} \sinh y - \frac{2}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} \\ &+ h \left[ \frac{1}{3} \sinh y + \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} \right] \\ &+ k \left[ \frac{1}{3} \sinh y + \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2}y\right) \sinh \frac{y}{2} + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}y\right) \cosh \frac{y}{2} \right] \end{aligned} \quad (118)$$

The hyperbolic functions of a tricomplex number  $x + hy + kz$  can then be expressed in terms of the elementary functions with the aid of the addition theorems Eqs. (113), (114) and of the expressions in Eqs. (115)-(118).

## 7 Tricomplex power series

A tricomplex series is an infinite sum of the form

$$a_0 + a_1 + a_2 + \cdots + a_l + \cdots, \quad (119)$$

where the coefficients  $a_n$  are tricomplex numbers. The convergence of the series (119) can be defined in terms of the convergence of its 3 real components. The convergence of a tricomplex series can however be studied using tricomplex variables. The main criterion for absolute convergence remains the comparison theorem, but this requires a number of inequalities which will be discussed further.

The modulus of a tricomplex number  $u = x + hy + kz$  can be defined as

$$|u| = (x^2 + y^2 + z^2)^{1/2}. \quad (120)$$

Since  $|x| \leq |u|, |y| \leq |u|, |z| \leq |u|$ , a property of absolute convergence established via a comparison theorem based on the modulus of the series (119) will ensure the absolute convergence of each real component of that series.

The modulus of the sum  $u_1 + u_2$  of the tricomplex numbers  $u_1, u_2$  fulfils the inequality

$$||u_1| - |u_2|| \leq |u_1 + u_2| \leq |u_1| + |u_2|. \quad (121)$$

For the product the relation is

$$|u_1 u_2| \leq \sqrt{3} |u_1| |u_2|, \quad (122)$$

which replaces the relation of equality extant for regular complex numbers. The equality in Eq. (122) takes place for  $x_1 = y_1 = z_1$  and  $x_2 = y_2 = z_2$ , i.e when both tricomplex numbers lie on the trisector line ( $t$ ). Using Eq. (93), the relation (122) can be written equivalently as

$$\frac{2}{3} \delta_1^2 \delta_2^2 + \frac{1}{3} \sigma_1^2 \sigma_2^2 \leq 3 \left( \frac{2}{3} \delta_1^2 + \frac{1}{3} \sigma_1^2 \right) \left( \frac{2}{3} \delta_2^2 + \frac{1}{3} \sigma_2^2 \right), \quad (123)$$

where  $\delta_j^2 = x_j^2 + y_j^2 + z_j^2 - x_j y_j - x_j z_j - y_j z_j, \sigma_j = x_j + y_j + z_j, j = 1, 2$ , the equality taking place for  $\delta_1 = 0, \delta_2 = 0$ . A particular form of Eq. (122) is

$$|u^2| \leq \sqrt{3} |u|^2, \quad (124)$$

and it can be shown that

$$|u^l| \leq 3^{(l-1)/2} |u|^l, \quad (125)$$

the equality in Eqs. (124) and (125) taking place for  $x = y = z$ . It can be shown from Eq. (95) that

$$|u^l|^2 = \frac{2}{3}\delta^{2l} + \frac{1}{3}\sigma^{2l}, \quad (126)$$

where  $\delta^2 = x^2 + y^2 + z^2 - xy - xz - yz$ ,  $\sigma = x + y + z$ . Then Eq. (125) can also be written as

$$\frac{2}{3}\delta^{2l} + \frac{1}{3}\sigma^{2l} \leq 3^{l-1} \left( \frac{2}{3}\delta^2 + \frac{1}{3}\sigma^2 \right)^l, \quad (127)$$

the equality taking place for  $\delta = 0$ . From Eqs. (122) and (125) it results that

$$|au^l| \leq 3^{l/2}|a||u|^l. \quad (128)$$

It can also be shown that

$$\left| \frac{1}{u} \right| \geq \frac{1}{|u|}, \quad (129)$$

the equality taking place for  $\sigma^2 = \delta^2$ , or  $xy + xz + yz = 0$ .

A power series of the tricomplex variable  $u$  is a series of the form

$$a_0 + a_1u + a_2u^2 + \cdots + a_lu^l + \cdots. \quad (130)$$

Since

$$\left| \sum_{l=0}^{\infty} a_lu^l \right| \leq \sum_{l=0}^{\infty} 3^{l/2}|a_l||u|^l, \quad (131)$$

a sufficient condition for the absolute convergence of this series is that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{3}|a_{l+1}||u|}{|a_l|} < 1. \quad (132)$$

Thus the series is absolutely convergent for

$$|u| < c_0, \quad (133)$$

where

$$c_0 = \lim_{l \rightarrow \infty} \frac{|a_l|}{\sqrt{3}|a_{l+1}|}. \quad (134)$$

The convergence of the series (130) can be also studied with the aid of a transformation which explicits the transverse and longitudinal parts of the variable  $u$  and of the constants  $a_l$ ,

$$x + hy + kz = v_1e_1 + \tilde{v}_1\tilde{e}_1 + v_+e_+, \quad (135)$$

where

$$v_1 = \frac{2x - y - z}{2}, \quad \tilde{v}_1 = \frac{\sqrt{3}}{2}(y - z), \quad v_+ = x + y + z, \quad (136)$$

and

$$e_1 = \frac{2 - h - k}{3}, \quad \tilde{e}_1 = \frac{h - k}{\sqrt{3}}, \quad e_+ = \frac{1 + h + k}{3}. \quad (137)$$

The variables  $v_1, \tilde{v}_1, v_+$  will be called the tricomplex canonical variables, and  $e_1, \tilde{e}_1, e_+$  will be called the tricomplex canonical base. In the geometric representation of Fig. 6,  $e_1, \tilde{e}_1$  are situated in the plane  $\Pi$ , and  $e_+$  is lying on the trisector line  $(t)$ . It can be checked that

$$e_1^2 = e_1, \quad \tilde{e}_1^2 = -e_1, \quad e_1\tilde{e}_1 = \tilde{e}_1, \quad e_1e_+ = 0, \quad \tilde{e}_1e_+ = 0, \quad e_+^2 = e_+. \quad (138)$$

The moduli of the bases in Eq. (138) are

$$|e_1| = \sqrt{\frac{2}{3}}, \quad |\tilde{e}_1| = \sqrt{\frac{2}{3}}, \quad |e_+| = \sqrt{\frac{1}{3}}, \quad (139)$$

and it can be checked that

$$|x + hy + kz|^2 = \frac{2}{3}(v_1^2 + \tilde{v}_1^2) + \frac{1}{3}v_+^2. \quad (140)$$

If  $u = u'u''$ , the transverse and longitudinal components are related by

$$v_1 = v'_1v''_1 - \tilde{v}'_1\tilde{v}''_1, \quad \tilde{v}_1 = v'_1\tilde{v}''_1 + \tilde{v}'_1v''_1, \quad v_+ = v'_+v''_+, \quad (141)$$

which show that, upon multiplication, the transverse components obey the same rules as the real and imaginary components of usual, two-dimensional complex numbers, and the rule for the longitudinal component is that of the regular multiplication of numbers.

If the constants in Eq. (130) are  $a_l = p_l + hq_l + kr_l$ , and

$$a_{l1} = \frac{2p_l - q_l - r_l}{2}, \quad \tilde{a}_{l1} = \frac{\sqrt{3}}{2}(q_l - r_l), \quad a_{l+} = p_l + q_l + r_l, \quad (142)$$

where  $p_0 = 1, q_0 = 0, r_0 = 0$ , the series (130) can be written as

$$\sum_{l=0}^{\infty} \left[ a_{l1}e_1 + \tilde{a}_{l1}\tilde{e}_1 \right] (v_1e_1 + \tilde{v}_1\tilde{e}_1)^l + e_+a_{l+}v_+^l. \quad (143)$$

The series in Eq. (143) is absolutely convergent for

$$|v_+| < c_+, \quad (v_1^2 + \tilde{v}_1^2)^{1/2} < c_1, \quad (144)$$

where

$$c_+ = \lim_{l \rightarrow \infty} \frac{|a_{l+}|}{|a_{l+1,u}|}, \quad c_1 = \lim_{l \rightarrow \infty} \frac{(a_{l1}^2 + \tilde{a}_{l1}^2)^{1/2}}{(a_{l+1,1}^2 + a_{l+1,2}^2)^{1/2}}. \quad (145)$$

The relations (144) and (140) show that the region of convergence of the series (130) is a cylinder of radius  $c_1 \sqrt{2/3}$  and height  $2c_+/\sqrt{3}$ , having the trisector line ( $t$ ) as axis and the origin as center, which can be called cylinder of convergence, as shown in Fig. 7.

It can be shown that  $c_1 = (1/\sqrt{3}) \min(c_+, c_1)$ , where  $\min$  designates the smallest of the numbers  $c_+, c_1$ . Using the expression of  $|u|$  in Eq. (138), it can be seen that the spherical region of convergence defined in Eqs. (133), (134) is a subset of the cylindrical region of convergence defined in Eqs. (144) and (145).

## 8 Analytic functions of tricomplex variables

The derivative of a function  $f(u)$  of the tricomplex variables  $u$  is defined as a function  $f'(u)$  having the property that

$$|f(u) - f(u_0) - f'(u_0)(u - u_0)| \rightarrow 0 \text{ as } |u - u_0| \rightarrow 0. \quad (146)$$

If the difference  $u - u_0$  is not parallel to one of the nodal hypersurfaces, the definition in Eq. (146) can also be written as

$$f'(u_0) = \lim_{u \rightarrow u_0} \frac{f(u) - f(u_0)}{u - u_0}. \quad (147)$$

The derivative of the function  $f(u) = u^m$ , with  $m$  an integer, is  $f'(u) = mu^{m-1}$ , as can be seen by developing  $u^m = [u_0 + (u - u_0)]^m$  as

$$u^m = \sum_{p=0}^m \frac{m!}{p!(m-p)!} u_0^{m-p} (u - u_0)^p, \quad (148)$$

and using the definition (146).

If the function  $f'(u)$  defined in Eq. (146) is independent of the direction in space along which  $u$  is approaching  $u_0$ , the function  $f(u)$  is said to be analytic, analogously to the case of functions of regular complex variables. [7] The function  $u^m$ , with  $m$  an integer, of the tricomplex variable  $u$  is analytic, because the difference  $u^m - u_0^m$  is always proportional to  $u - u_0$ , as can be seen from Eq. (148). Then series of integer powers of  $u$  will also be analytic

functions of the tricomplex variable  $u$ , and this result holds in fact for any commutative algebra.

If an analytic function is defined by a series around a certain point, for example  $u = 0$ , as

$$f(u) = \sum_{k=0}^{\infty} a_k u^k, \quad (149)$$

an expansion of  $f(u)$  around a different point  $a$ ,

$$f(u) = \sum_{k=0}^{\infty} c_k (u - a)^k, \quad (150)$$

can be obtained by substituting in Eq. (149) the expression of  $u^k$  according to Eq. (148). Assuming that the series are absolutely convergent so that the order of the terms can be modified and ordering the terms in the resulting expression according to the increasing powers of  $u - a$  yields

$$f(u) = \sum_{k,l=0}^{\infty} \frac{(k+l)!}{k!l!} a_{k+l} a^l (u - a)^k. \quad (151)$$

Since the derivative of order  $k$  at  $u = a$  of the function  $f(u)$ , Eq. (149), is

$$f^{(k)}(a) = \sum_{l=0}^{\infty} \frac{(k+l)!}{l!} a_{k+l} a^l, \quad (152)$$

the expansion of  $f(u)$  around  $u = a$ , Eq. (151), becomes

$$f(u) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a) (u - a)^k, \quad (153)$$

which has the same form as the series expansion of the usual 2-dimensional complex functions. The relation (153) shows that the coefficients in the series expansion, Eq. (150), are

$$c_k = \frac{1}{k!} f^{(k)}(a). \quad (154)$$

The rules for obtaining the derivatives and the integrals of the basic functions can be obtained from the series of definitions and, as long as these series expansions have the same form as the corresponding series for the 2-dimensional complex functions, the rules of derivation and integration remain unchanged.

If the tricomplex function  $f(u)$  of the tricomplex variable  $u$  is written in terms of the real functions  $F(x, y, z), G(x, y, z), H(x, y, z)$  of real variables  $x, y, z$  as

$$f(u) = F(x, y, z) + hG(x, y, z) + kH(x, y, z), \quad (155)$$



then relations of equality exist between partial derivatives of the functions  $F, G, H$ . These relations can be obtained by writing the derivative of the function  $f$  as

$$\frac{1}{\Delta x + h\Delta y + k\Delta z} \left[ \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial y} \Delta y + \frac{\partial F}{\partial z} \Delta z + h \left( \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y + \frac{\partial G}{\partial z} \Delta z \right) + k \left( \frac{\partial H}{\partial x} \Delta x + \frac{\partial H}{\partial y} \Delta y + \frac{\partial H}{\partial z} \Delta z \right) \right], \quad (156)$$

where the difference appearing in Eq. (98) is  $u - u_0 = \Delta x + h\Delta y + k\Delta z$ . The relations between the partials derivatives of the functions  $F, G, H$  are obtained by setting successively in Eq. (156)  $\Delta x \rightarrow 0, \Delta y = 0, \Delta z = 0$ ; then  $\Delta x = 0, \Delta y \rightarrow 0, \Delta z = 0$ ; and  $\Delta x = 0, \Delta y = 0, \Delta z \rightarrow 0$ . The relations are

$$\frac{\partial F}{\partial x} = \frac{\partial G}{\partial y}, \quad \frac{\partial G}{\partial x} = \frac{\partial H}{\partial y}, \quad \frac{\partial H}{\partial x} = \frac{\partial F}{\partial y}, \quad (157)$$

$$\frac{\partial F}{\partial x} = \frac{\partial H}{\partial z}, \quad \frac{\partial G}{\partial x} = \frac{\partial F}{\partial z}, \quad \frac{\partial H}{\partial x} = \frac{\partial G}{\partial z}, \quad (158)$$

$$\frac{\partial G}{\partial y} = \frac{\partial H}{\partial z}, \quad \frac{\partial H}{\partial y} = \frac{\partial F}{\partial z}, \quad \frac{\partial F}{\partial y} = \frac{\partial G}{\partial z}. \quad (159)$$

The relations (157)-(159) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from Eqs. (157)-(159) that the components  $F$  solutions of the equations

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y \partial z} = 0, \quad \frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x \partial z} = 0, \quad \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial x \partial y} = 0, \quad (160)$$

$$\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial y \partial z} = 0, \quad \frac{\partial^2 G}{\partial y^2} - \frac{\partial^2 G}{\partial x \partial z} = 0, \quad \frac{\partial^2 G}{\partial z^2} - \frac{\partial^2 G}{\partial x \partial y} = 0, \quad (161)$$

$$\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial y \partial z} = 0, \quad \frac{\partial^2 H}{\partial y^2} - \frac{\partial^2 H}{\partial x \partial z} = 0, \quad \frac{\partial^2 H}{\partial z^2} - \frac{\partial^2 H}{\partial x \partial y} = 0. \quad (162)$$

It can also be shown that the differences  $F - G, F - H, G - H$  are solutions of the equation of Laplace,

$$\Delta(F - G) = 0, \quad \Delta(F - H) = 0, \quad \Delta(G - H) = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (163)$$

If a geometric transformation is considered in which to a point  $u$  is associated the point  $f(u)$ , it can be shown that the tricomplex function  $f(u)$  transforms a straight line parallel to the trisector line  $(t)$  in a straight line parallel to  $(t)$ , and transforms a plane parallel to the nodal plane  $\Pi$  in a plane parallel to  $\Pi$ . A transformation generated by a tricomplex function  $f(u)$  does not conserve in general the angle of intersecting lines.

## 9 Integrals of tricomplex functions

The singularities of tricomplex functions arise from terms of the form  $1/(u-a)^m$ , with  $m > 0$ . Functions containing such terms are singular not only at  $u = a$ , but also at all points of a plane  $(\Pi_a)$  through the point  $a$  and parallel to the nodal plane  $\Pi$  and at all points of a straight line  $(t_a)$  passing through  $a$  and parallel to the trisector line  $(t)$ .

The integral of a tricomplex function between two points  $A, B$  along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free from singularities is zero,

$$\oint_{\Gamma} f(u) du = 0, \quad (164)$$

where it is supposed that a surface  $S$  spanning the closed loop  $\Gamma$  is not intersected by any of the planes and is not threaded by any of the lines associated with the singularities of the function  $f(u)$ . Using the expression, Eq. (155) for  $f(u)$  and the fact that  $du = dx + hdy + kdz$ , the explicit form of the integral in Eq. (164) is

$$\oint_{\Gamma} f(u) du = \oint_{\Gamma} [Fdx + Hdy + Gdz + h(Gdx + Fdy + Hdz) + k(Hdx + Gdy + Fdz)]. \quad (165)$$

If the functions  $F, G, H$  are regular on a surface  $S$  spanning the loop  $\Gamma$ , the integral along the loop  $\Gamma$  can be transformed with the aid of the theorem of Stokes in an integral over the surface  $S$  of terms of the form  $\partial H/\partial x - \partial F/\partial y$ ,  $\partial G/\partial x - \partial F/\partial z$ ,  $\partial G/\partial y - \partial H/\partial z, \dots$  which are equal to zero by Eqs. (157)-(159), and this proves Eq. (164).

If there are singularities on the surface  $S$ , the integral  $\oint f(u) du$  is not necessarily equal to zero. If  $f(u) = 1/(u-a)$  and the loop  $\Gamma_a$  is situated in the half-space above the plane  $(\Pi_a)$  and encircles once the line  $(t_a)$ , then

$$\oint_{\Gamma_a} \frac{du}{u-a} = \frac{2\pi}{\sqrt{3}}(h-k). \quad (166)$$

This is due to the fact that the integral of  $1/(u-a)$  along the loop  $\Gamma_a$  is equal to the integral of  $1/(u-a)$  along a circle  $(C_a)$  with the center on the line  $(t_a)$  and perpendicular to this line, as shown in Fig. 8.

$$\oint_{\Gamma_a} \frac{du}{u-a} = \oint_{C_a} \frac{du}{u-a}, \quad (167)$$

this being a corollary of Eq. (164). The integral on the right-hand side of Eq. (167) can be evaluated with the aid of the trigonometric form Eq. (90) of the tricomplex quantity  $u - a$ , so that

$$\frac{du}{u - a} = \frac{h - k}{\sqrt{3}} d\phi, \quad (168)$$

which by integration over  $d\phi$  from 0 to  $2\pi$  yields Eq. (166).

The integral  $\oint_{\Gamma_a} du(u - a)^m$ , with  $m$  an integer number not equal to -1, is equal to zero, because  $\int du(u - a)^m = (u - a)^{m+1}/(m + 1)$ , and  $(u - a)^{m+1}/(m + 1)$  is singlevalued,

$$\oint_{\Gamma_a} du(u - a)^m = 0, \text{ for } m \text{ integer, } m \neq -1. \quad (169)$$

If  $f(u)$  is an analytic tricomplex function which can be expanded in a series as written in Eq. (150), and the expansion holds on the curve  $\Gamma$  and on a surface spanning  $\Gamma$ , then from Eqs. (168) and (169) it follows that

$$\oint_{\Gamma} \frac{f(u)du}{u - a} = \frac{2\pi}{\sqrt{3}}(h - k)f(a). \quad (170)$$

Substituting in the right-hand side of Eq. (170) the expression of  $f(u)$  in terms of the real components  $F, G, H$ , Eq. (155), at  $u = a$ , yields

$$\oint_{\Gamma} \frac{f(u)du}{u - a} = \frac{2\pi}{\sqrt{3}}[H - G + h(F - H) + k(G - F)]. \quad (171)$$

Since the sum of the real components in the paranthesis from the right-hand side of Eq. (171) is equal to zero, this equation determines only the differences between the components  $F, G, H$ . If  $f(u)$  can be expanded as written in Eq. (150) on  $\Gamma$  and on a surface spanning  $\Gamma$ , then from Eqs. (166) and (169) it also results that

$$\oint_{\Gamma} \frac{f(u)du}{(u - a)^{m+1}} = \frac{2\pi}{\sqrt{3}m!}(h - k)f^{(m)}(a), \quad (172)$$

where the fact that has been used that the derivative  $f^{(m)}(a)$  of order  $m$  of  $f(u)$  at  $u = a$  is related to the expansion coefficient in Eq. (150) according to Eq. (154). The relation (172) can also be obtained by successive derivations of Eq. (170).

If a function  $f(u)$  is expanded in positive and negative powers of  $u - u_j$ , where  $u_j$  are fourcomplex constants,  $j$  being an index, the integral of  $f$  on a closed loop  $\Gamma$  is determined by the terms in the expansion of  $f$  which are of the form  $a_j/(u - u_j)$ ,

$$f(u) = \cdots + \sum_j \frac{a_j}{u - u_j} + \cdots. \quad (173)$$

In Eq. (173),  $u_j$  is the pole and  $a_j$  is the residue relative to the pole  $u_j$ . Then the integral of  $f$  on a closed loop  $\Gamma$  is

$$\oint_{\Gamma} f(u)du = \frac{2\pi}{\sqrt{3}}(h-k) \sum_j \text{int}(u_{j\Pi}, \Gamma_{\Pi})a_j, \quad (174)$$

where the functional  $\text{int}(M, C)$ , defined for a point  $M$  and a closed curve  $C$  in a two-dimensional plane, is given by

$$\text{int}(M, C) = \begin{cases} 1 & \text{if } M \text{ is an interior point of } C, \\ 0 & \text{if } M \text{ is exterior to } C \end{cases} \quad (175)$$

and  $u_{j\Pi}, \Gamma_{\Pi}$  are the projections of the point  $u_j$  and of the curve  $\Gamma$  on the nodal plane  $\Pi$ , as shown in Fig. 9.

## 10 Factorization of tricomplex polynomials

A polynomial of degree  $m$  of the tricomplex variable  $u = x + hy + kz$  has the form

$$P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m, \quad (176)$$

where the constants are in general tricomplex numbers,  $a_l = p_l + hq_l + kr_l$ ,  $l = 1, \dots, m$ . In order to write the polynomial  $P_m(u)$  as a product of factors, the variable  $u$  and the constants  $a_l$  will be written in the form which explicits the transverse and longitudinal components,

$$P_m(u) = \sum_{l=0}^m (a_{l1}e_1 + \tilde{a}_{l1}\tilde{e}_1)(v_1e_1 + \tilde{v}_1\tilde{e}_1)^{m-l} + e_+ \sum_{l=0}^m a_{l+} v_+^{m-l}, \quad (177)$$

where the constants have been defined previously in Eq. (142). Due to the properties in Eq. (138), the transverse part of the polynomial  $P_m(u)$  can be written as a product of linear factors of the form

$$\sum_{l=0}^m (a_{l1}e_1 + \tilde{a}_{l1}\tilde{e}_1)(v_1e_1 + \tilde{v}_1\tilde{e}_1)^{m-l} = \prod_{l=1}^m [(v_1 - v_{l1})e_1 + (\tilde{v}_1 - \tilde{v}_{l1})\tilde{e}_1], \quad (178)$$

where the quantities  $v_{l1}, \tilde{v}_{l1}$  are real numbers. The longitudinal part of  $P_m(u)$ , Eq. (177), can be written as a product of linear or quadratic factors with real coefficients, or as a product of linear factors which, if imaginary, appear always in complex conjugate pairs. Using the latter form for the simplicity of notations, the longitudinal part can be written as

$$\sum_{l=0}^m a_{l+} v_+^{m-l} = \prod_{l=1}^m (v_+ - v_{l+}), \quad (179)$$

where the quantities  $v_{l+}$  appear always in complex conjugate pairs. Due to the orthogonality of the transverse and longitudinal components, Eq. (138), the polynomial  $P_m(u)$  can be written as a product of factors of the form

$$P_m(u) = \prod_{l=1}^m [(v_1 - v_{l1})e_1 + (\tilde{v}_1 - \tilde{v}_{l1})\tilde{e}_1 + (v_+ - v_{l+})e_+]. \quad (180)$$

These relations can be written with the aid of Eqs. (135) as

$$P_m(u) = \prod_{l=1}^m (u - u_l), \quad (181)$$

where

$$u_l = v_{l1}e_1 + \tilde{v}_{l1}\tilde{e}_1 + v_{l+}e_+. \quad (182)$$

The roots  $v_{l+}$  and the roots  $v_{l1}e_1 + \tilde{v}_{l1}\tilde{e}_1$  defined in Eq. (178) may be ordered arbitrarily. This means that Eq. (182) gives sets of  $m$  roots  $u_1, \dots, u_m$  of the polynomial  $P_m(u)$ , corresponding to the various ways in which the roots  $v_{l+}, v_{l1}e_1 + \tilde{v}_{l1}\tilde{e}_1$  are ordered according to  $l$  in each group. Thus, while the tricomplex components in Eq. (177) taken separately have unique factorizations, the polynomial  $P_m(q)$  can be written in many different ways as a product of linear factors.

If  $P(u) = u^2 - 1$ , the degree is  $m = 2$ , the coefficients of the polynomial are  $a_1 = 0, a_2 = -1$ , the coefficients defined in Eq. (142) are  $a_{21} = -1, a_{22} = 0, a_{2u} = -1$ . The expression of  $P(u)$ , Eq. (177), is  $(e_1v_1 + \tilde{e}_1\tilde{v}_1)^2 - e_1 + e_+(v_+^2 - 1)$ . The factorizations in Eqs. (178) and (179) are  $(e_1v_1 + \tilde{e}_1\tilde{v}_1)^2 - e_1 = [e_1(v_1 + 1) + \tilde{e}_1\tilde{v}_1][e_1(v_1 - 1) + \tilde{e}_1\tilde{v}_1]$  and  $v_+^2 - 1 = (v_+ + 1)(v_+ - 1)$ . The factorization of  $P(u)$ , Eq. (181), is  $P(u) = (u - u_1)(u - u_2)$ , where according to Eq. (182) the roots are  $u_1 = \pm e_1 \pm e_+, u_2 = -v_1$ . If  $e_1$  and  $e_+$  are expressed with the aid of Eq. (137) in terms of  $h$  and  $k$ , the factorizations of  $P(u)$  are obtained as

$$u^2 - 1 = (u + 1)(u - 1), \quad (183)$$

or as

$$u^2 - 1 = \left(u + \frac{1 - 2h - 2k}{3}\right) \left(u - \frac{1 - 2h - 2k}{3}\right). \quad (184)$$

It can be checked that  $(\pm e_1 \pm e_+)^2 = e_1 + e_+ = 1$ .

## 11 Representation of tricomplex complex numbers by irreducible matrices

If the matrix in Eq. (34) representing the tricomplex number  $u$  is called  $U$ , and

$$T = \begin{pmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (185)$$

which is the matrix appearing in Eq. (17), it can be checked that

$$TUT^{-1} = \begin{pmatrix} x - \frac{y+z}{2} & \frac{\sqrt{3}}{2}(y-z) & 0 \\ -\frac{\sqrt{3}}{2}(y-z) & x - \frac{y+z}{2} & 0 \\ 0 & 0 & x+y+z \end{pmatrix}. \quad (186)$$

The relations for the variables  $x - (y+z)/2$ ,  $(\sqrt{3}/2)(y-z)$  and  $x+y+z$  for the multiplication of tricomplex numbers have been written in Eqs. (26), (28) and (29). The matrices  $TUT^{-1}$  provide an irreducible representation [8] of the tricomplex numbers  $u = x + hy + kz$ , in terms of matrices with real coefficients.

## 12 Conclusions

The operations of addition and multiplication of the tricomplex numbers introduced in this work have a simple geometric interpretation based on the amplitude  $\rho$ , polar angle  $\theta$  and azimuthal angle  $\phi$ . An exponential form exists for the tricomplex numbers, and a trigonometric form exists involving the variables  $\rho, \theta$  and  $\phi$ . The tricomplex functions defined by series of powers are analytic, and the partial derivatives of the components of the tricomplex functions are closely related. The integrals of tricomplex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the tricomplex numbers depends on the cyclic variable  $\phi$  leads to the concept of pole and residue for integrals on closed paths. The polynomials of tricomplex variables can be written as products of linear or quadratic factors.

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## FIGURE CAPTIONS

Fig. 1. Nodal plane  $\Pi$ , of equation  $x + y + z = 0$ , and trisector line  $(t)$ , of equation  $x = y = z$ , both passing through the origin  $O$  of the rectangular axes  $x, y, z$ .

Fig. 2. Tricomplex variables  $s, d, \theta, \phi$  for the tricomplex number  $x + hy + kz$ , represented by the point  $P(x, y, z)$ . The azimuthal angle  $\phi$  is shown in the plane parallel to  $\Pi$ , passing through  $P$ , which intersects the trisector line  $(t)$  at  $Q$  and the axes of coordinates  $x, y, z$  at the points  $A, B, C$ . The orthogonal axes  $\xi_1^\parallel, \xi_2^\parallel, \xi_3^\parallel$  have the origin at  $Q$ .

Fig. 3. Invariant circle for the multiplication of tricomplex numbers, lying in a plane perpendicular to the trisector line and passing through the points  $(1,0,0)$ ,  $(0,1,0)$  and  $(0,0,1)$ . The center of the circle is at the point  $(1/3, 1/3, 1/3)$ , and its radius is  $\sqrt{2/3}$ .

Fig. 4. Surfaces of constant  $\rho$ , which are surfaces of rotation having the trisector line  $(t)$  as axis.

Fig. 5. Graphs of the cosexponential functions  $cx, mx, px$ .

Fig. 6. Unit vectors  $e_1, \tilde{e}_1, e_+$  of the orthogonal system of coordinates with origin at  $Q$ . The plane parallel to  $\Pi$  passing through  $P$  intersects the trisector line  $(t)$  at  $Q$  and the axes of coordinates  $x, y, z$  at the points  $A, B, C$ .

Fig. 7. Cylinder of convergence of tricomplex series, of radius  $c_1\sqrt{2/3}$  and height  $2c_+/\sqrt{3}$ , having the axis parallel to the trisector line.

Fig. 8. The integral of  $1/(u - a)$  along the loop  $\Gamma_a$  is equal to the integral of  $1/(u - a)$  along a circle  $(C_a)$  with the center on the line  $(t_a)$  and perpendicular to this line.



Fig. 9. Integration path  $\Gamma$ , pole  $u_j$  and their projections  $\Gamma_\Pi, u_{j\Pi}$  on the nodal plane  $\Pi$ .

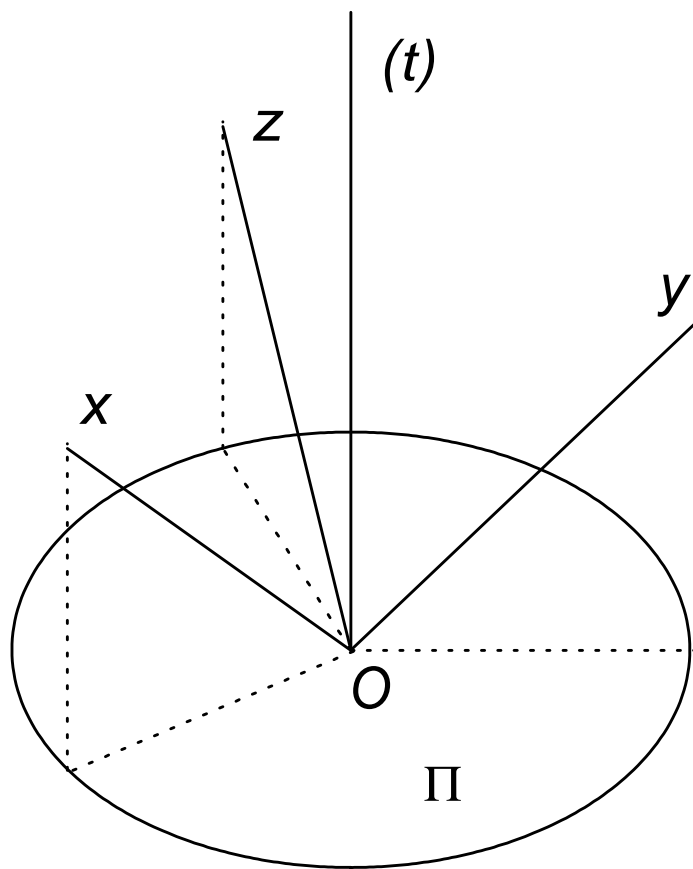


Fig. 1

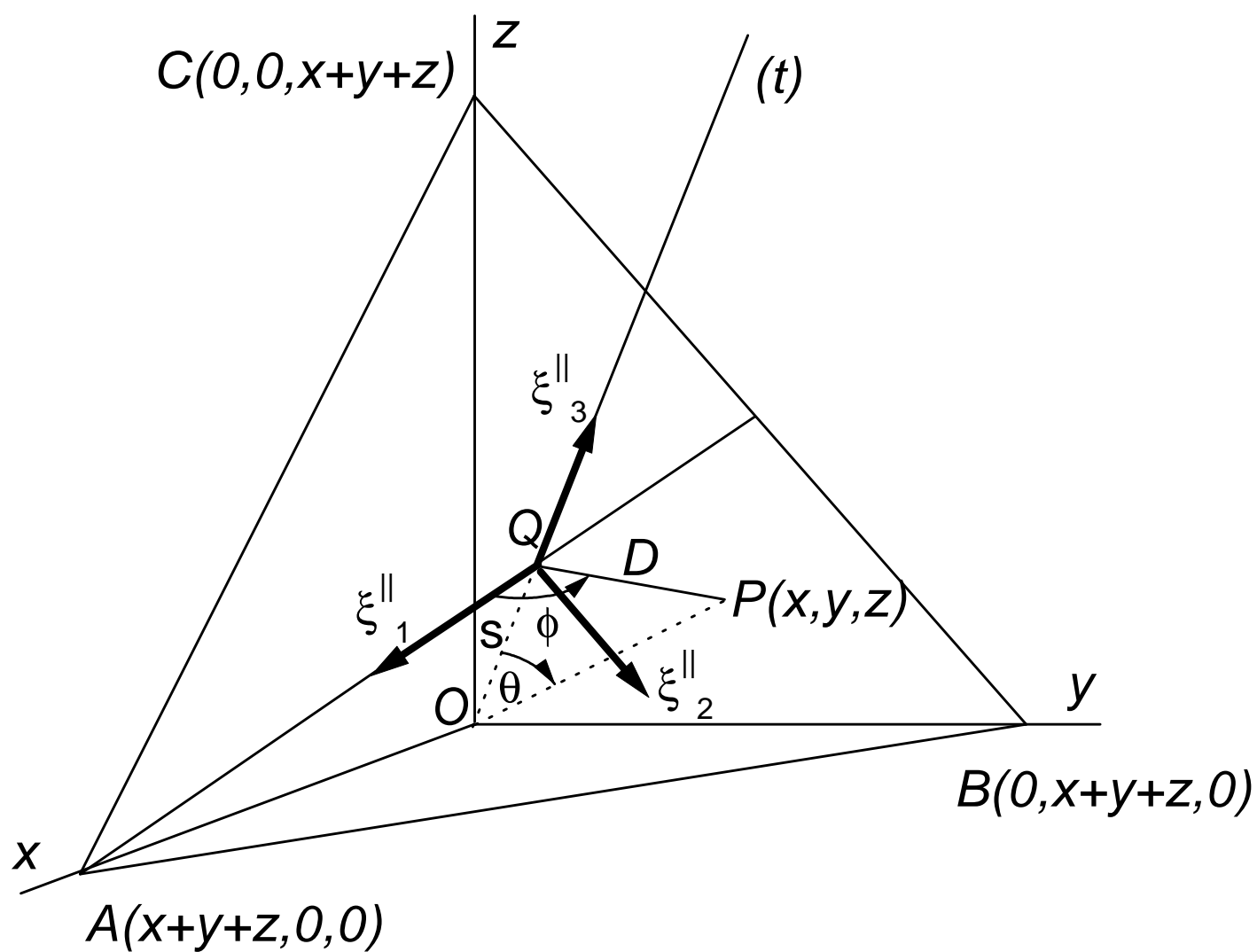


Fig. 2

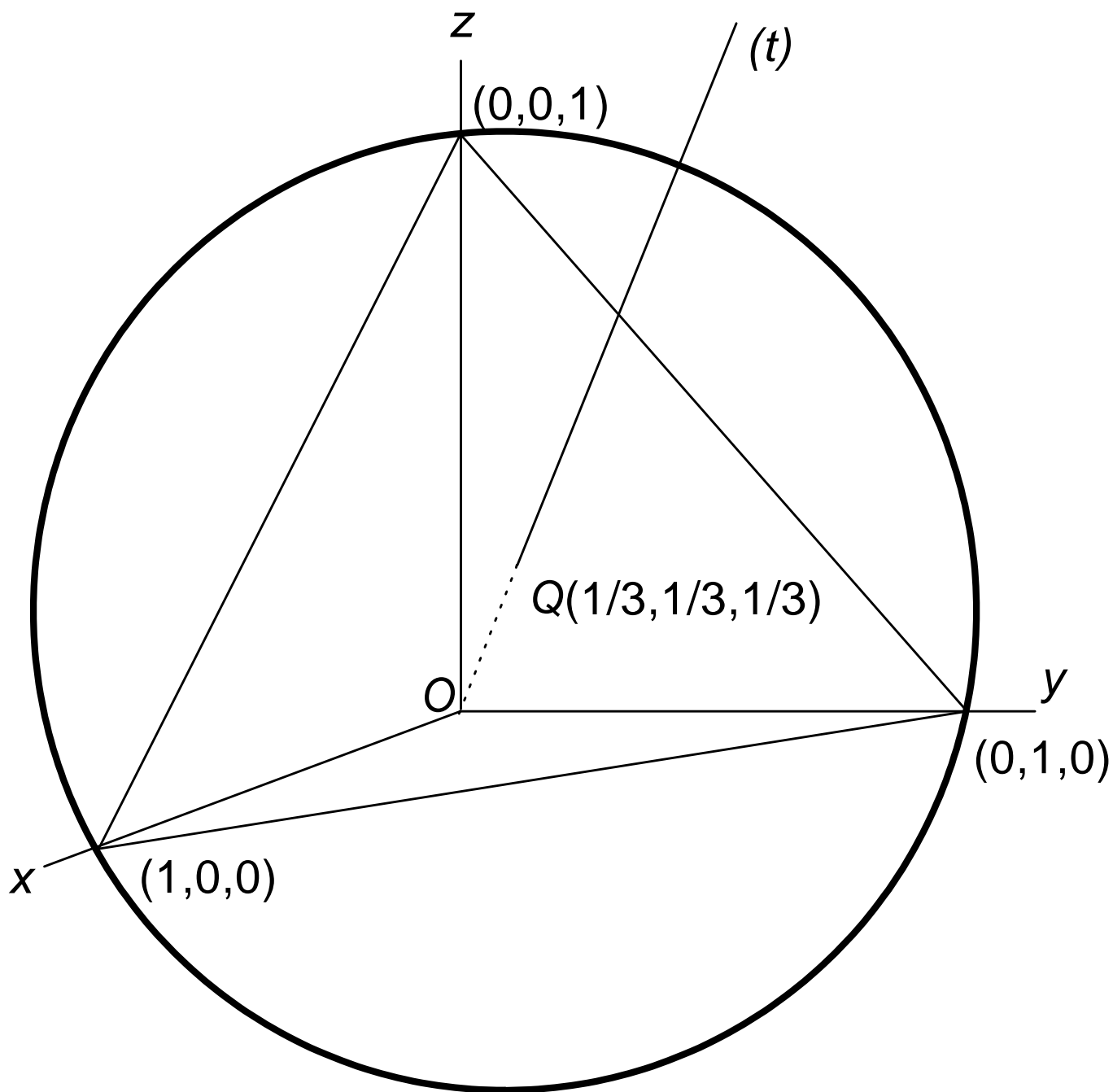


Fig. 3

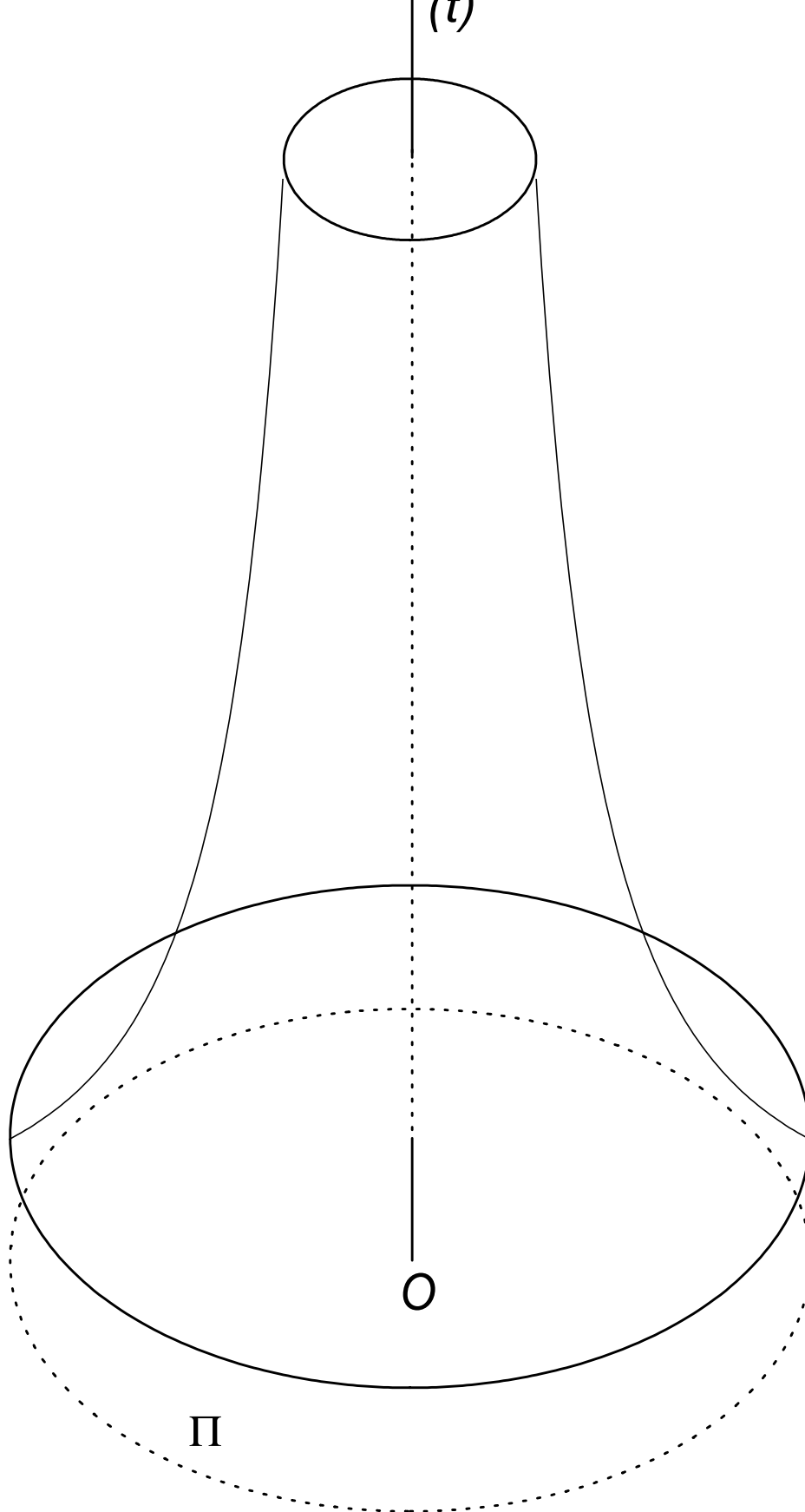
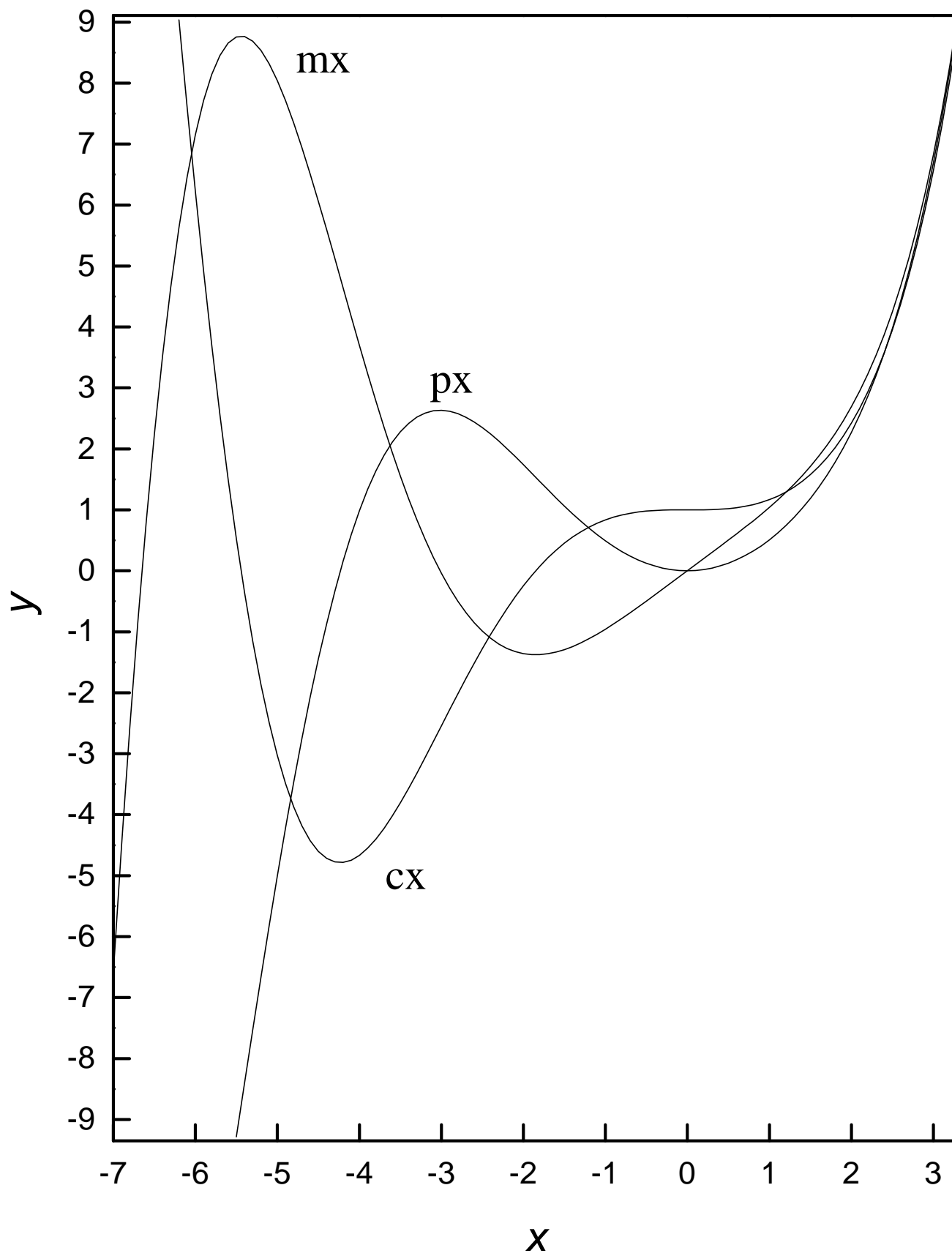


Fig. 4

Fig. 5



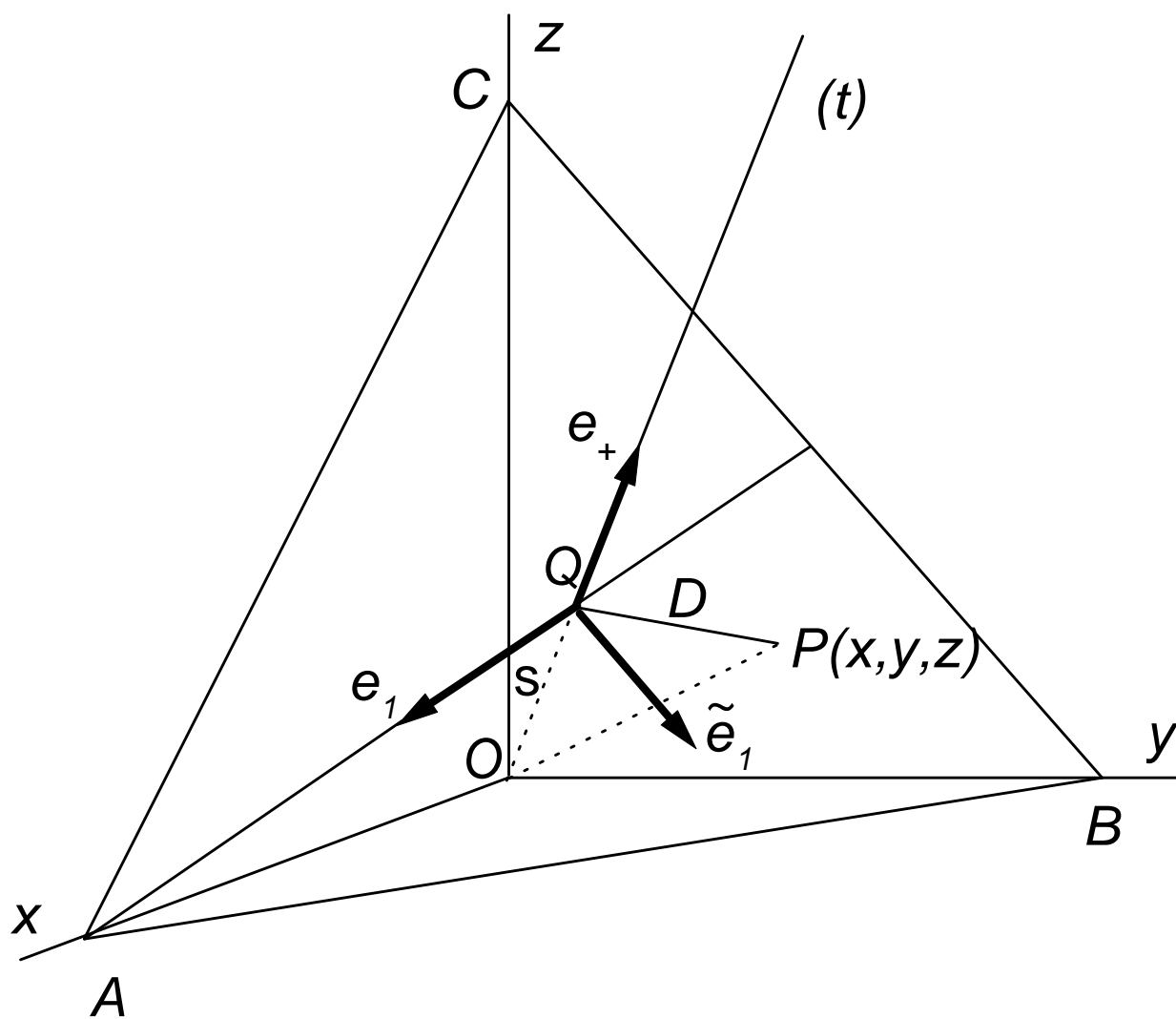


Fig. 6

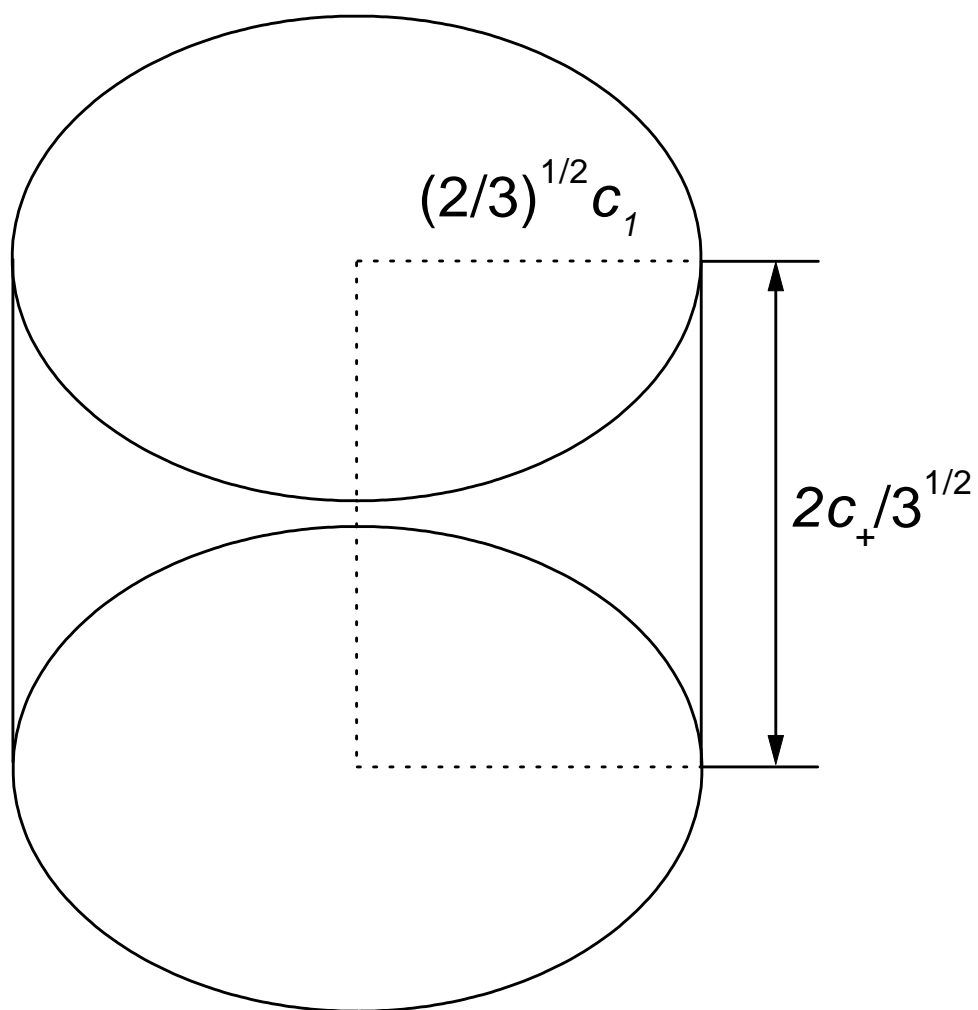


Fig. 7



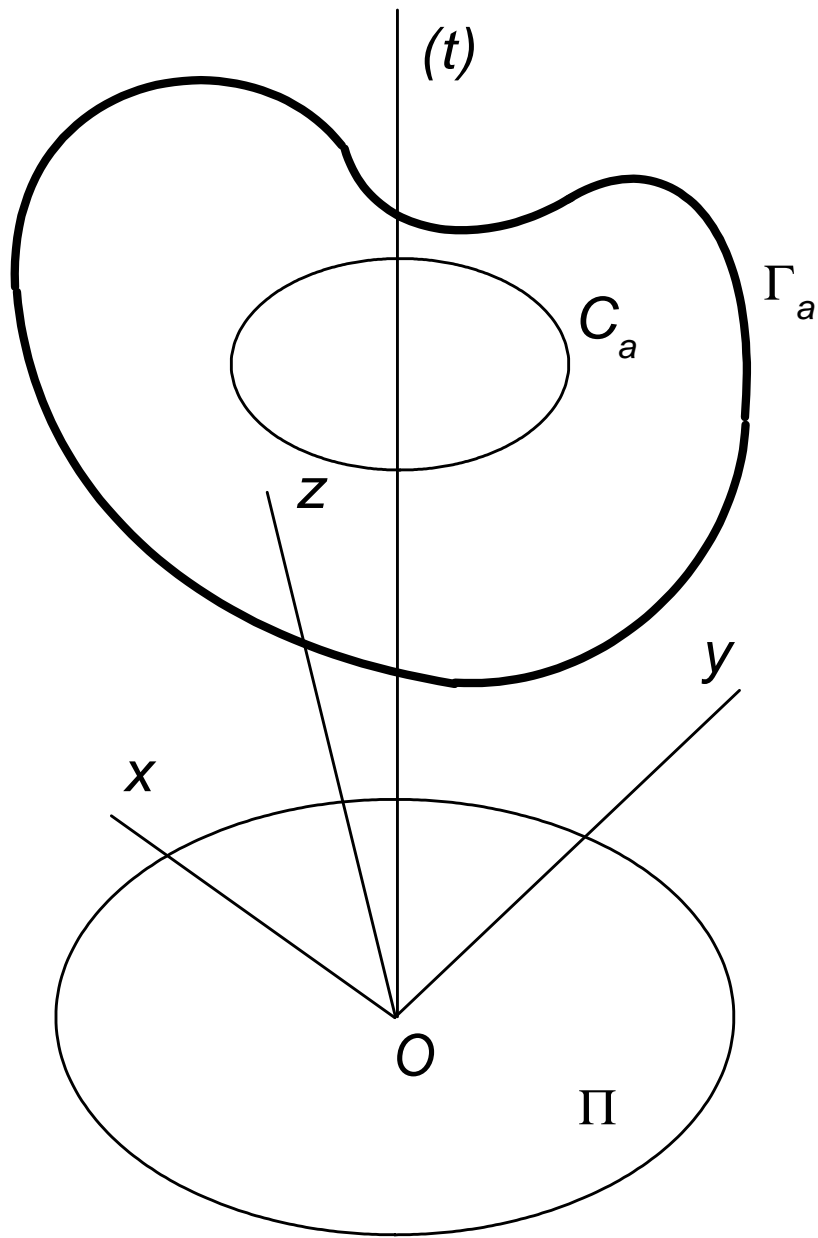


Fig. 8

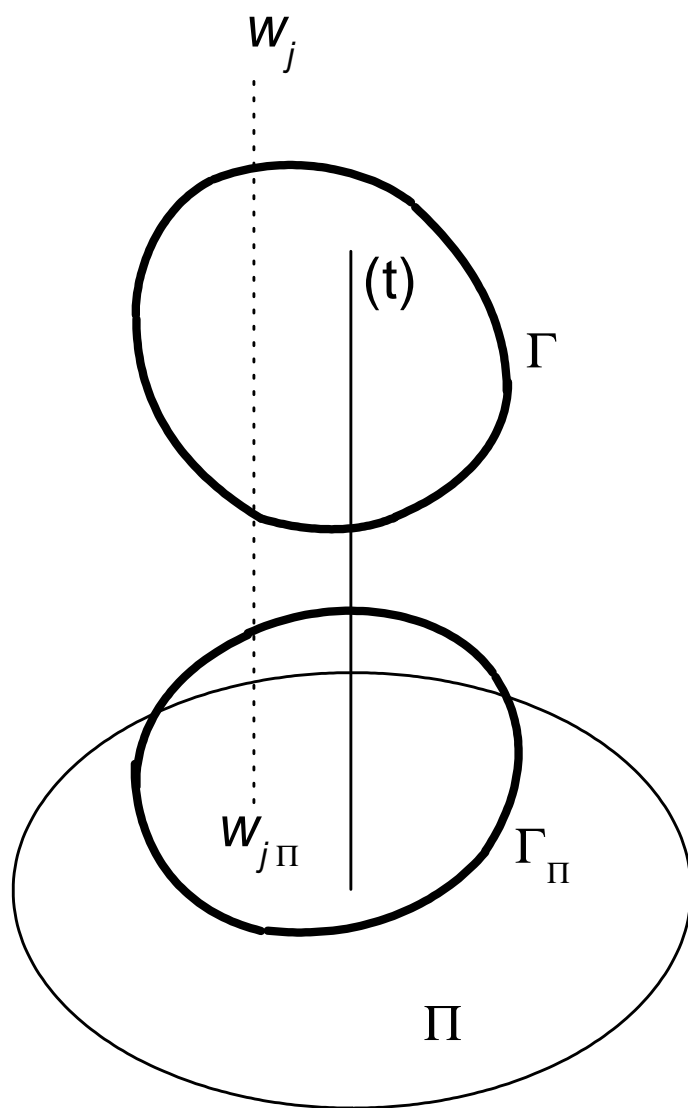


Fig. 9